UNIVERSITÀ DEGLI STUDI DI BERGAMO | Dipartimento di Ingegneria e Scienze Applicate

Entropy-stable modal DG for scale-resolving flow simulations

WHO

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WHERE

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WHEN

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Scope

Present recent developments in a high-fidelity CFD solver for turbulent flows, carried out with a view to future GPU acceleration for industrial-scale cases

Outline

- The underlying numerical method
  - Discontinuous Galerkin (DG) discretization of the governing equations
  - Change of variables and entropy framework
  - Entropy projection and Direct Enforcement of the Entropy Balance
- Considerations on the Scale Resolving Simulation (SRS) of turbulent flows
- A simple wall-model applied to implicit LES
- Conclusion and future work

Approximation of the numerical solution

In DG method the solution is approximated as

$$\mathbf{q}(\mathbf{x},t)|_K = \mathbf{\Phi}(\mathbf{x})^\top \mathbf{Q}(t) = \sum_{i=1}^{N_{Dof}^k} Q_i(t)\phi_i(\mathbf{x})$$

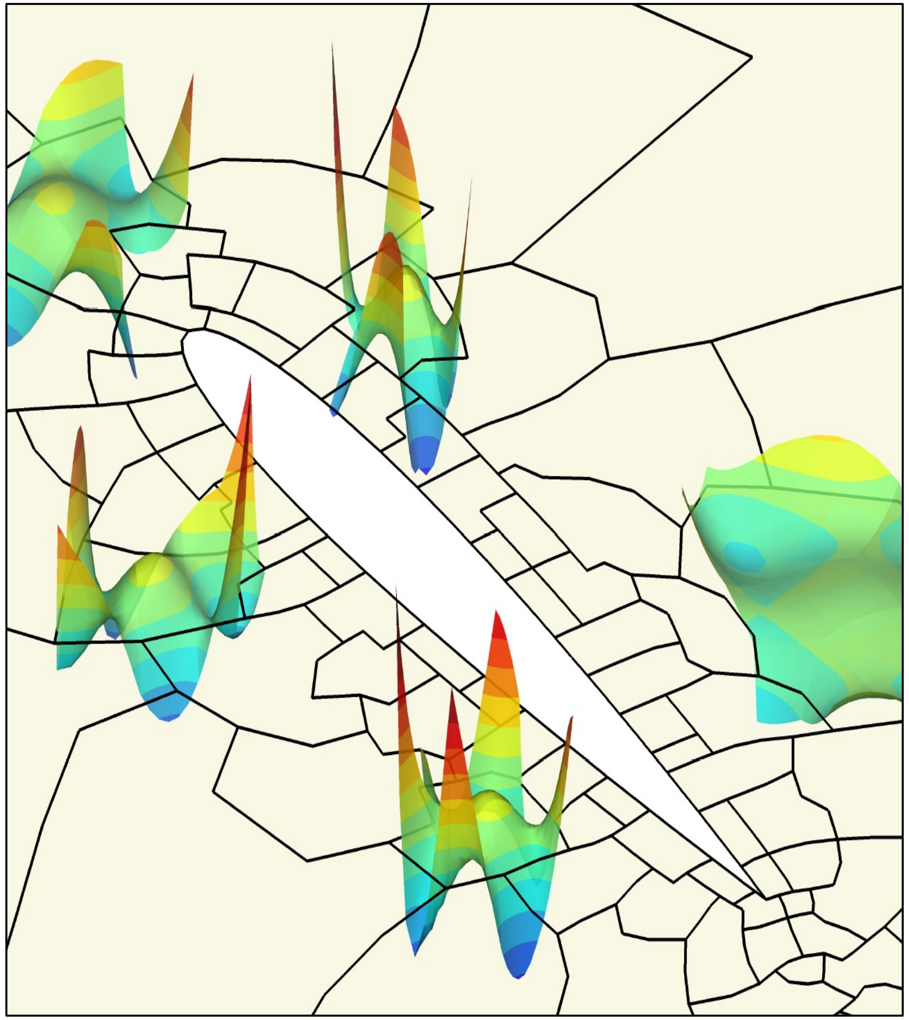
on a mesh $\mathcal{K}_h = \{K\}$ and where $N_{Dof}^k = \mathrm{card}(\mathbb{P}_d^k)$

We define discrete polynomial spaces in physical (mesh) coordinates

$$\mathbb{P}_d^k(\mathcal{K}_h) \stackrel{\text{def}}{=} \{\phi \in L^2(\Omega) \mid \phi_{|T} \in \mathbb{P}_d^k(K), \forall K \in \mathcal{K}_h\}$$

A modal set of orthonormal and hierarchical basis functions is obtained from monomials via the Modified Gram-Schmidt procedure *[1]*

The basis can be defined for elements of any shape



[1] Bassi, F., Botti, L., Colombo, A., Di Pietro, D. A., and Tesini, P. On the flexibility of agglomeration based physical space discontinuous Galerkin discretizations. JCP (2012).

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DG discretization of the Navier-Stokes (NS) equations

The system of governing equations can be written in compact form as

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}_{\mathrm{c},i}(\mathbf{q})}{\partial x_i} + \frac{\partial \mathbf{F}_{\mathrm{v},i}(\mathbf{q}, \nabla \mathbf{q})}{\partial x_i} = \mathbf{b}(\mathbf{q})$$

where $\mathbf{q} = [\rho, \rho\mathbf{u}, \rho E]$ is the set of conservative variables, $\mathbf{F}_{\mathrm{c}}$ and $\mathbf{F}_{\mathrm{v}}$ are the convective and viscous fluxes, $\mathbf{b}(\mathbf{q}) = [0, \mathbf{g}, \mathbf{u}^{\top}\mathbf{g}]$ is the gravitational source term and $i = \{1, \dots, 2 + d\}$

Alternative sets of variables, $\mathbf{w}$, have been investigated by several authors

$$\mathbf{P}(\mathbf{w}) \equiv \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}_{\mathrm{c},i}(\mathbf{w})}{\partial x_i} + \frac{\partial \mathbf{F}_{\mathrm{v},i}(\mathbf{w}, \nabla \mathbf{w})}{\partial x_i} = \mathbf{b}(\mathbf{w})$$

for a number of reasons

- deal with low Mach number flows $[p, \mathbf{u}, T]$ *[2]*
- ensure the positivity of thermodynamics variables at a discrete level $[\ln(p), \mathbf{u}, \ln(T)]$
- to guarantee entropy conservation/stability at the discrete level *[3,4,5]*
- ….

[2] F. Bassi, C. De Bartolo, R. Hartmann, A. Nigro. "A discontinuous Galerkin method for inviscid low Mach number flows" (2009) J Comput Phys 2009;228(11):3996–4011, DOI:10.1016/j.jcp.2009.02.021
[3] Hughes, T.J.R., Franca, L.P., Mallet, M. "A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics" (1986) Computer Methods in Applied Mechanics and Engineering, 54 (2), DOI: 10.1016/0045-7825(86)90127-1
[4] Diosady, L.T., Murman, S.M. Tensor-product preconditioners for higher-order space–time discontinuous Galerkin methods (2017) Journal of Computational Physics, 330, DOI: 10.1016/j.jcp.2016.11.022
[5] Colombo, A., Crivellini, A., Nigro, A. "On the entropy conserving/stable implicit DG discretization of the Euler equations in entropy variables" (2022) Computers and Fluids, DOI: 10.1016/j.compfluid.2021.105198

DG discretization of the Navier-Stokes (NS) equations

Alternative sets of variables, $\mathbf{w}$, have been investigated by several authors

$$\mathbf{P}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{F}_{\mathrm{c},i}(\mathbf{w})}{\partial x_i} + \frac{\partial \mathbf{F}_{\mathrm{v},i}(\mathbf{w}, \nabla \mathbf{w})}{\partial x_i} = \mathbf{b}(\mathbf{w})$$

By multiplying by an arbitrary test function $\mathbf{\Phi} = \{\phi_1, \ldots, \phi_{2+d}\}$ and integrating by parts, we obtain the weak formulation

$$\int_\Omega \mathbf{\Phi}^\top \mathbf{P}(\mathbf{w})\frac{\partial \mathbf{w}}{\partial t} d\Omega - \int_\Omega \left(\frac{\partial \mathbf{\Phi}}{\partial x_i}\right)^\top \left[\mathbf{F}_{\mathrm{c},i}(\mathbf{w}) - \mathbf{F}_{\mathrm{v},i}(\mathbf{w}, \nabla \mathbf{w})\right] d\Omega$$
$$+ \int_{\partial\Omega} \mathbf{\Phi}^\top \left[\mathbf{F}_{\mathrm{c},i}(\mathbf{w}) + \mathbf{F}_{\mathrm{v},i}(\mathbf{w}, \nabla \mathbf{w})\right] n_i d\sigma - \int_\Omega \mathbf{\Phi}^\top \mathbf{b}(\mathbf{w}) d\Omega = 0$$

solution and test function are replaced with DG approximations belonging to $\left[\mathbb{P}_d^k(\mathcal{K}_h)\right]^{d+2}$

DG discretization of the Navier-Stokes (NS) equations

The discretization consists in seeking, the elements of the vector **W** of unknown Dof s.t.

$$\sum_{K\in\mathcal{K}_h}\int_K \phi_i P_{l,\kappa}\left(\mathbf{w}_h\right)\phi_j \frac{dW_{\kappa,j}}{dt}\,d\Omega$$

$$-\sum_{K\in\mathcal{K}_h}\int_K \frac{\partial\phi_i}{\partial x_n}\left[F_{\mathrm{c},l,n}\left(\mathbf{w}_h\right)+F_{\mathrm{v},l,n}\left(\mathbf{w}_h,\nabla\mathbf{w}_h+\mathbf{r}\left(\llbracket\mathbf{w}_h\rrbracket\right)\right)\right]d\Omega$$

$$+\sum_{F\in\mathcal{F}_h}\int_F \llbracket\phi_i\rrbracket\left[\widehat{F}_{\mathrm{c},l}\left(\mathbf{w}_h^{\pm},\mathbf{n}_F\right)+\widehat{F}_{\mathrm{v},l}\left(\mathbf{w}_h^{\pm},\left(\nabla\mathbf{w}_h+\eta_F\mathbf{r}_F\left(\llbracket\mathbf{w}_h\rrbracket\right)\right)^{\pm},\mathbf{n}_F\right)\right]d\sigma$$

$$-\sum_{K\in\mathcal{K}_h}\int_K \phi_i b_l\left(\mathbf{w}_h\right)d\Omega=0$$

repeated indices imply summation $\kappa=1,\dots,d+2$, $l=1,\dots,N_{Dof}^{K}$, $n=1,\dots,d$. Interface convective fluxes are treated with exact *[7]* or approximated Riemann solvers (possibly entropy conserving/stable *[6,9]*) while the viscous fluxes contribution follows *[8]*

[6] F. Ismail and P. Roe. Affordable, entropy-consistent Euler flux functions II: Entropy production at shocks, Journal of Computational Physics, 228 (15): 5410–5436, (2009)
[7] J. Gottlieb and C. Groth. Assessment of Riemann solvers for unsteady one-dimensional inviscid flows of perfect gases. JCP, 78(2): 437–58, (1988)
[8] F. Bassi, and S. Rebay. A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations. J.CP (1997)
[9] E. Tadmor. Entropy stability theory for difference approximations of nonlinear conservation laws and related time-dependent problems, Acta Numerica, 12: 451–512, (2003)
[10] Worku, Z. A., Hicken, J. E., & Zingg, D. W. Very high-order symmetric positive-interior quadrature rules on triangles and tetrahedra. J. Comput. Appl. Math. (2026).

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$$-\sum_{K\in\mathcal{K}_h}\int_K \frac{\partial\phi_i}{\partial x_n}\left[F_{\mathrm{c},l,n}\left(\mathbf{w}_h\right)+F_{\mathrm{v},l,n}\left(\mathbf{w}_h,\nabla\mathbf{w}_h+\mathbf{r}\left(\llbracket\mathbf{w}_h\rrbracket\right)\right)\right]d\Omega$$

$$+\sum_{F\in\mathcal{F}_h}\int_F \llbracket\phi_i\rrbracket\left[\widehat{F}_{\mathrm{c},l}\left(\mathbf{w}_h^{\pm},\mathbf{n}_F\right)+\widehat{F}_{\mathrm{v},l}\left(\mathbf{w}_h^{\pm},\left(\nabla\mathbf{w}_h+\eta_F\mathbf{r}_F\left(\llbracket\mathbf{w}_h\rrbracket\right)\right)^{\pm},\mathbf{n}_F\right)\right]d\sigma$$

$$-\sum_{K\in\mathcal{K}_h}\int_K \phi_i b_l\left(\mathbf{w}_h\right)d\Omega=0$$

integrals are computed with Gaussian and/or PIQuad *[10]* rules able to exactly evaluate the polynomial term $\int_K \mathbf{\Phi}^\top\mathbf{\Phi}\, d\Omega$

repeated indices imply summation $\kappa=1,\dots,d+2$, $l=1,\dots,N_{Dof}^K$, $n=1,\dots,d$. Interface convective fluxes are treated with exact *[7]* or approximated Riemann solvers (possibly entropy conserving/stable *[6,9]*) while the viscous fluxes contribution follows *[8]*

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$$\mathbf{M_P}(\mathbf{W})\frac{d\mathbf{W}}{dt} + \mathbf{R}(\mathbf{W}) = \mathbf{0}$$

$$- \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial\phi_i}{\partial x_n}\left[F_{c,l,n}\left(\mathbf{w}_h\right) + F_{v,l,n}\left(\mathbf{w}_h, \nabla\mathbf{w}_h + \mathbf{r}\left([\![\mathbf{w}_h]\!]\right)\right)\right] d\Omega$$

$$+ \sum_{F\in\mathcal{F}_h}\int_F [\![\phi_i]\!]\left[\widehat{F}_{c,l}\left(\mathbf{w}_h^{\pm}, \mathbf{n}_F\right) + \widehat{F}_{v,l}\left(\mathbf{w}_h^{\pm}, \left(\nabla\mathbf{w}_h + \eta_F\mathbf{r}_F\left([\![\mathbf{w}_h]\!]\right)\right)^{\pm}, \mathbf{n}_F\right)\right] d\sigma$$

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$$\mathbf{M_P}(\mathbf{W})\frac{d\mathbf{W}}{dt} + \mathbf{R}(\mathbf{W}) = \mathbf{0}$$

- $\mathbf{M_P}(\mathbf{W})$ is a block diagonal (element-wise) operator and needs to be inverted also for explicit time integration, its inversion scales as k^{3d}
- Thanks to the orthonormality of shapes functions by using conservative variables ($\mathbf{cv}$), $\mathbf{w} = \mathbf{q} \;\rightarrow\; \mathbf{M_P} = \mathbf{I}$ for cells of any shape!
- In our experience, a solver based on conservative variables is less robust than one based on primitive variables, possibly logarithmic, or entropy variables ($\mathbf{ev}$)
- Entropy variables ($\mathbf{w} = \mathbf{v}$) allow conservation/dissipation of entropy at discrete level *[11]*
- Time integration can be also crucial for conservation properties *[12]*, it would deserve a dedicated talk…

[11] Hughes, T.J.R., Franca, L.P., Mallet, M. "A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics" (1986) Computer Methods in Applied Mechanics and Engineering, 54 (2), DOI: 10.1016/0045-7825(86)90127-1

[12] A. Colombo, A. Crivellini and A. Nigro. On the entropy conserving/stable implicit DG discretization of the Euler equations in entropy variables. C&F, 232: 105198, (2022)

Entropy variables and entropy projection

We want to combine the advantages of **cv** in terms of efficiency, with those of **ev** in terms of robustness and physical accuracy

Entropy variables symmetrize the governing equations and are related to an entropy function $\mathcal{S}$ globally conserved for smooth solutions of the Euler equations

$$\mathcal{S} = -\frac{\rho s}{\gamma - 1} \qquad s = \ln \frac{p}{\rho^\gamma} \qquad \mathbf{v} = \frac{\partial \mathcal{S}}{\partial \mathbf{q}} = \left[\frac{\gamma - s}{\gamma - 1} - \frac{\rho}{2p}|\mathbf{u}|^2, \frac{\rho \mathbf{u}}{p}, -\frac{\rho}{p}\right]^\top$$

Hughes et al. firstly proposed the use of this set for the solution of Navier-Stokes equations *[3]*

Using entropy projection (**cv2ev**) *[13,14]* the solution is sought for the conservative variables ($\mathbf{M_P} = \mathbf{I}$), but spatial discretization is computed from $\mathbf{v}_h^*$, the L2-projection of **cv** on **ev**

$$\int_K \mathbf{\Phi}_h^\top \mathbf{v}_h^* d\Omega = \int_K \mathbf{\Phi}_h^\top \mathbf{v}\left(\mathbf{q}_h\right) d\Omega$$

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[3] Hughes, T.J.R., Franca, L.P., Mallet, M. "A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics" (1986) Computer Methods in Applied Mechanics and Engineering, 54 (2), DOI: 10.1016/0045-7825(86)90127-1
[13] J. Chan, H. Ranocha, A. M. Rueda-Ramírez, G. Gassner and T. Warburton. On the entropy projection and the robustness of high order entropy stable discontinuous Galerkin schemes for under-resolved flows. Frontiers in Physics, 356, (2022)
[14] G. Gassner, A. Winters and D. Kopriva. Split form nodal discontinuous Galerkin schemes with summation-by-parts property for the compressible Euler equations. JCP, 327: 39-66, (2016)

Entropy variables and entropy projection

Using entropy projection the weak formulation reads

$$\int_\Omega \mathbf{\Phi}^\top \frac{\partial \mathbf{q}}{\partial t} d\Omega - \int_\Omega \left(\frac{\partial \mathbf{\Phi}}{\partial x_i} \right)^\top \left[\mathbf{F}_{c,i}(\mathbf{v}^*{}_h) - \mathbf{F}_{v,i}(\mathbf{v}^*{}_h, \nabla \mathbf{v}^*{}_h) \right] d\Omega$$

$$+ \int_{\partial\Omega} \mathbf{\Phi}^\top \left[\mathbf{F}_{c,i}(\mathbf{v}^*{}_h) + \mathbf{F}_{v,i}(\mathbf{v}^*{}_h, \nabla \mathbf{v}^*{}_h) \right] n_i d\sigma - \int_\Omega \mathbf{\Phi}^\top \mathbf{b}(\mathbf{v}^*{}_h) d\Omega = 0$$

…let us focus on the convective terms and replace $\mathbf{\Phi}^\top$ with $\mathbf{v}_h^{*\top}$ (same polynomial space!)

$$\sum_{K \in \mathcal{K}_h} \int_K \mathbf{v}_h^{*\top} \frac{\partial \mathbf{q}_h}{\partial t} d\Omega - \sum_{K \in \mathcal{K}_h} \int_K \frac{\partial \mathbf{v}_h^{*\top}}{\partial x_i} \mathbf{F}_{c,i} \left(\mathbf{v}_h^* \right) d\Omega$$

$$+ \sum_{F \in \mathcal{F}_h} \int_F [\![\mathbf{v}_h^{*\top}]\!] \widehat{\mathbf{F}}_c \left((\mathbf{v}_h^*)^\pm, \mathbf{n}_F \right) d\sigma = 0$$

Entropy variables and entropy projection

...let us focus on the convective terms and replace $\mathbf{\Phi}^\top$ with $\mathbf{v}_h^{*\top}$ (same polynomial space!)

$$\sum_{K\in\mathcal{K}_h}\int_K \mathbf{v}_h^{*\top}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega - \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathbf{v}_h^{*\top}}{\partial x_i}\mathbf{F}_{c,i}\left(\mathbf{v}_h^*\right)d\Omega + \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{v}_h^{*\top}]\!]\widehat{\mathbf{F}}_c\left((\mathbf{v}_h^*)^\pm, \mathbf{n}_F\right)d\sigma = 0$$

by definition and applying the divergence theorem...

$$\sum_{K\in\mathcal{K}_h}\int_K \mathbf{v}_h^{*\top}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega = \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathcal{S}\left(\mathbf{q}_h\right)}{\partial \mathbf{q}_h}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega$$

$$\sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathbf{v}_h^{*\top}}{\partial x_i}\mathbf{F}_{c,i}\left(\mathbf{v}_h^*\right)d\Omega = \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \psi_i\left(\mathbf{v}_h^*\right)}{\partial x_i}d\Omega = \sum_{F\in\mathcal{F}_h}\int_F [\![\psi_i\left(\mathbf{v}_h^*\right)]\!]n_{F,i}d\sigma$$

where $\boldsymbol{\psi} = \rho\mathbf{u}$ we obtain

$$\sum_{K\in\mathcal{K}_h}\int_K \frac{d\mathcal{S}\left(\mathbf{q}_h\right)}{dt}d\Omega + \sum_{F\in\mathcal{F}_h}\int_F \underbrace{\left[[\![\mathbf{v}_h^{*\top}]\!]\widehat{\mathbf{F}}_c\left((\mathbf{v}_h^*)^\pm, \mathbf{n}_F\right) - [\![\psi_i(\mathbf{v}_h^*)]\!]n_{F,i}\right]}_{\geq 0 \text{ if } \widehat{\mathbf{F}} \text{ is entropy conserving/stable}}d\sigma = 0$$

Entropy variables and entropy projection

…let us focus on the convective terms and replace $\boldsymbol{\Phi}^\top$ with $\mathbf{v}_h^{*\top}$ (same polynomial space!)

$$\sum_{K\in\mathcal{K}_h}\int_K \mathbf{v}_h^{*\top}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega - \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathbf{v}_h^{*\top}}{\partial x_i}\mathbf{F}_{c,i}\left(\mathbf{v}_h^*\right)d\Omega + \sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{v}_h^{*\top}]\!]\widehat{\mathbf{F}}_c\left(\left(\mathbf{v}_h^*\right)^{\pm},\mathbf{n}_F\right)d\sigma = 0$$

by definition and applying the divergence theorem…at the discrete level…

$$\sum_{K\in\mathcal{K}_h}\int_K \mathbf{v}_h^{*\top}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega = \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathcal{S}\left(\mathbf{q}_h\right)}{\partial \mathbf{q}_h}\frac{\partial \mathbf{q}_h}{\partial t}d\Omega$$

$$\sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \mathbf{v}_h^{*\top}}{\partial x_i}\mathbf{F}_{c,i}\left(\mathbf{v}_h^*\right)d\Omega = \sum_{K\in\mathcal{K}_h}\int_K \frac{\partial \psi_i\left(\mathbf{v}_h^*\right)}{\partial x_i}d\Omega \neq \sum_{F\in\mathcal{F}_h}\int_F [\![\psi_i\left(\mathbf{v}_h^*\right)]\!]n_{F,i}d\sigma$$

where $\boldsymbol{\psi} = \rho\mathbf{u}$, we obtain

$$\sum_{K\in\mathcal{K}_h}\int_K \frac{d\mathcal{S}\left(\mathbf{q}_h\right)}{dt}d\Omega \gtrless 0$$

A POSSIBLE FIX: “*over-integration*”
use of quadrature rules with an extremely high degree of exactness, not known a priori,
with a dramatic degradation of performance

Direct Enforcement of the Entropy Balance (DEEB)

To avoid over-integration the DEEB proposed by Abgrall *[15]* and applied to DG by Abgrall et al. *[16]* and Chen and Shu *[17]* is used

$$\sum_{K\in\mathcal{K}_h}\int_K \mathbf{\Phi}_h^\top \frac{\partial \mathbf{q}_h}{\partial t} d\Omega - \sum_{K\in\mathcal{K}_h}\int_K \left(\frac{\partial \mathbf{\Phi}_h}{\partial x_i}\right)^\top \mathbf{F}_{c,i}\left(\mathbf{v}_h^*\right) d\Omega$$

$$+\sum_{F\in\mathcal{F}_h}\int_F [\![\mathbf{\Phi}_h^\top]\!]\widehat{\mathbf{F}}_c\left((\mathbf{v}_h^*)^\pm, \mathbf{n}_F\right) d\sigma + \sum_{K\in\mathcal{K}_h} \alpha_K \frac{\int_K \mathbf{\Phi}_h^\top (\mathbf{v}_h^* - \mathbf{v}_{h,0}^*) d\Omega}{\int_K (\mathbf{v}_h^* - \mathbf{v}_{h,0}^*)^\top (\mathbf{v}_h^* - \mathbf{v}_{h,0}^*) d\Omega} = 0$$

where $\mathbf{v}_{h,0}^*$ is the elemental mean value of $\mathbf{v}_h^*$

DEEB was proposed for a general framework and can be used with any set of variables, but when dealing with $\mathbf{ev}$ it only compensates for the integration error

$$\alpha_K = \int_K \left(\frac{\partial \mathbf{v}_h^*}{\partial x_i}\right)^\top \mathbf{F}_{c,i}\left(\mathbf{w}_h\right) d\Omega - \sum_{F\in\mathcal{F}_K}\int_F [\![\psi_i\left(\mathbf{w}_h\right)]\!] n_{F,i} d\sigma$$

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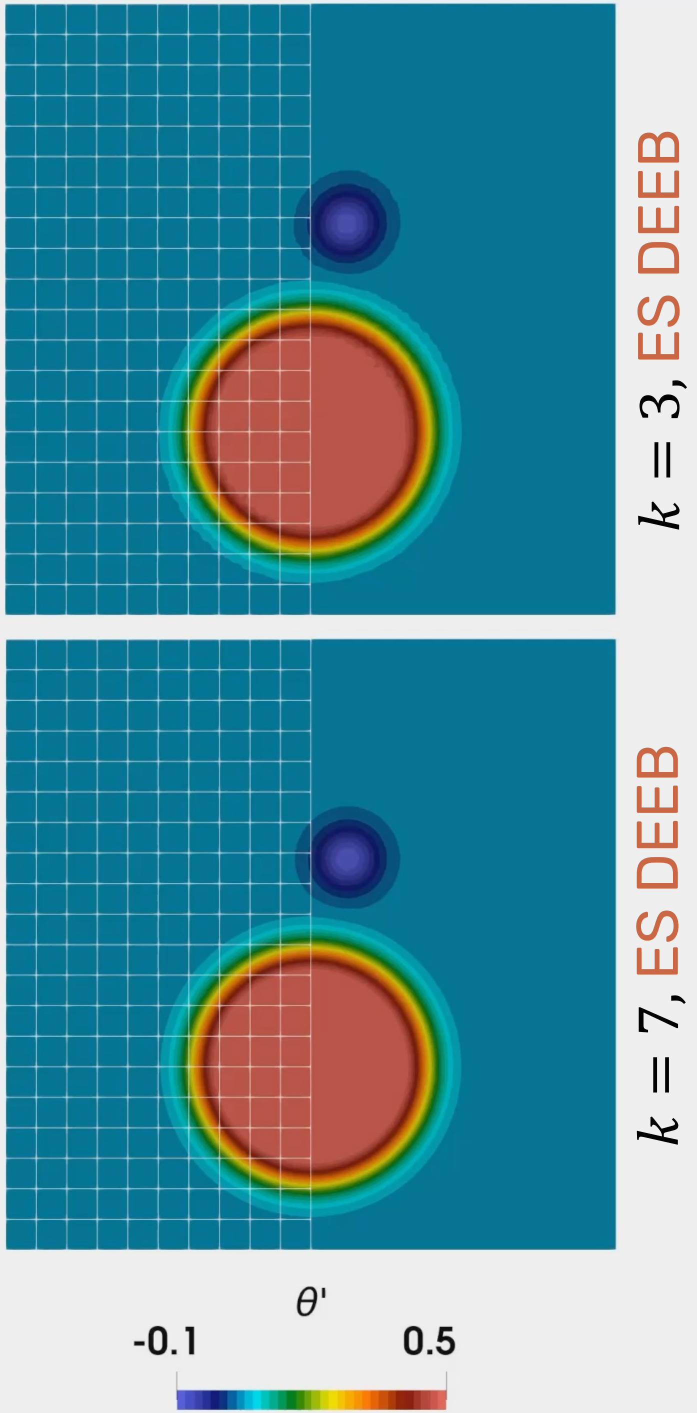
[16] R. Abgrall, P. Öffner and H. Ranocha. Reinterpretation and extension of entropy correction terms for residual distribution and discontinuous Galerkin schemes: application to structure preserving discretization. JCP, 453: 110955, (2022)
[17] T. Chen and C.W. Shu. Review of entropy stable discontinuous Galerkin methods for systems of conservation laws on unstructured simplex meshes. CSIAM, 1(1): 1–52, (2020)

Colliding thermal bubbles

Two air bubbles, one warmer and the other colder than the environment collide, mix, and move upward *[18]*

- Euler equations, $k = \{3, \dots, 7\}$ Entropy Conserving (EC) and Entropy Stable (ES)
- 20×20 quads.
- RK—SSP 3-5, $\mathrm{CFL} = 0.1$
- Simulation end time $t_f = 600s$
- NOI: "standard", degree-of-exactness (doe) for quadrature rules set to $2k + 1$
- OI: "over-integrated", $doe = 30$
- DEEB: correction term with $doe = 2k + 1$



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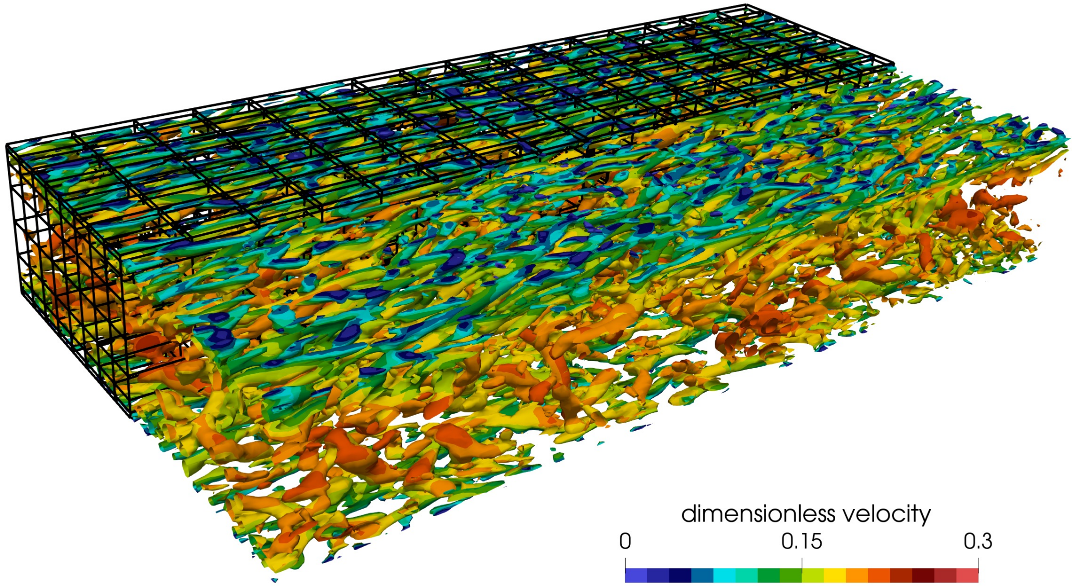
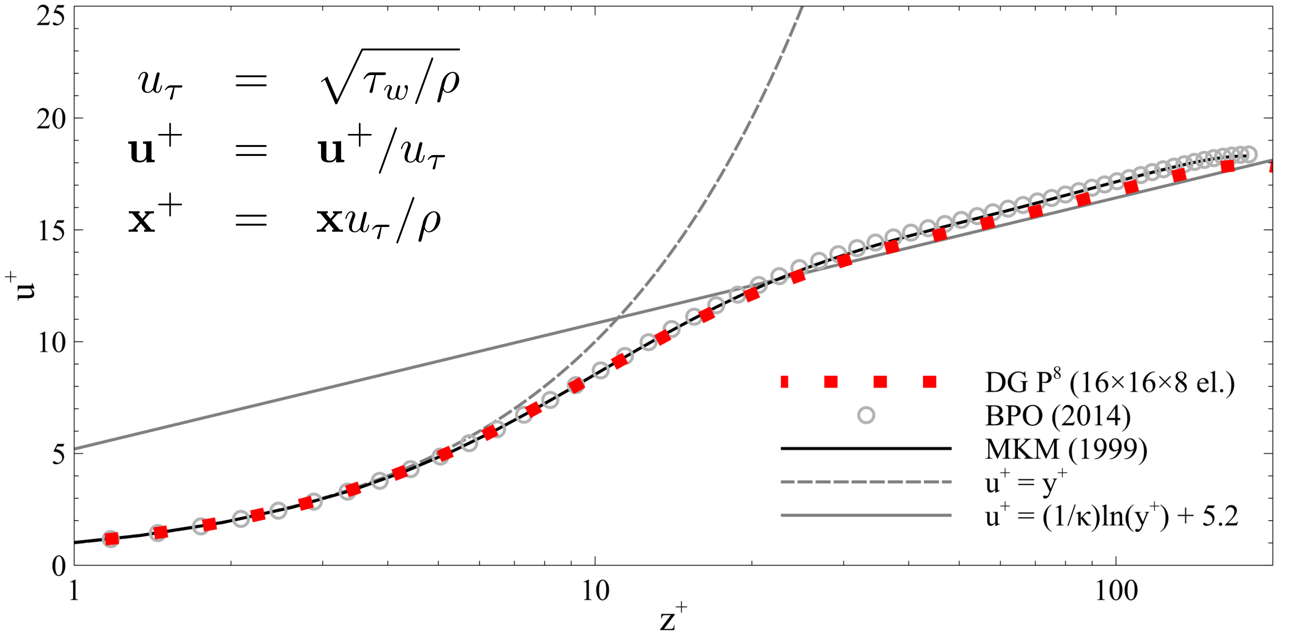
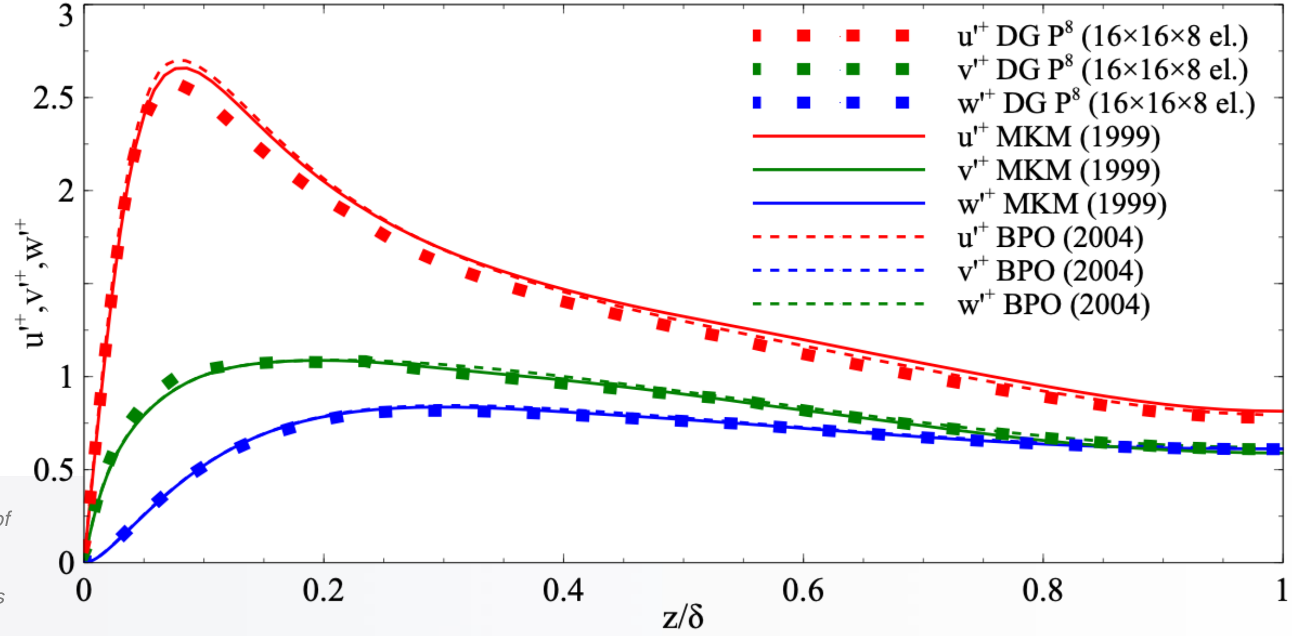
EC *Ismail-Roe*		k = 3	4	5	6	7
cv2ev$^{\mathrm{OI}}$	s	600	600	600	600	600
	$\varepsilon(s)$	■	■	■	■	■
cv2ev$^{\mathrm{NOI}}$	s	417	600	426	600	471
	$\varepsilon(s)$	×	10^{-9}	×	10^{-9}	×
cv2ev$^{\mathrm{DEEB}}$	s	600	600	600	600	600
	$\varepsilon(s)$	■	■	■	■	■
ev$^{\mathrm{OI}}$	s	600	600	600	600	600
	$\varepsilon(s)$	■	■	■	■	■
ev$^{\mathrm{NOI}}$	s	417	600	426	600	471
	$\varepsilon(s)$	×	10^{-9}	×	10^{-9}	×
ev$^{\mathrm{DEEB}}$	s	600	600	600	600	600
	$\varepsilon(s)$	■	■	■	■	10^{-11}
cv$^{\mathrm{OI}}$	s	292	362	291	366	308
	$\varepsilon(s)$	×	×	×	×	×
cv$^{\mathrm{NOI}}$	s	274	342	275	340	297
	$\varepsilon(s)$	×	×	×	×	×

[18] A. Robert, "Bubble convection experiments with a semi-implicit formulation of the Euler equations," Journal of Atmospheric Sciences, vol. 50, no. 13, pp. 1865 – 1873, 1993

implicit-LES - channel flow $\mathrm{Re}_\tau = \{180, 395, 590\}$



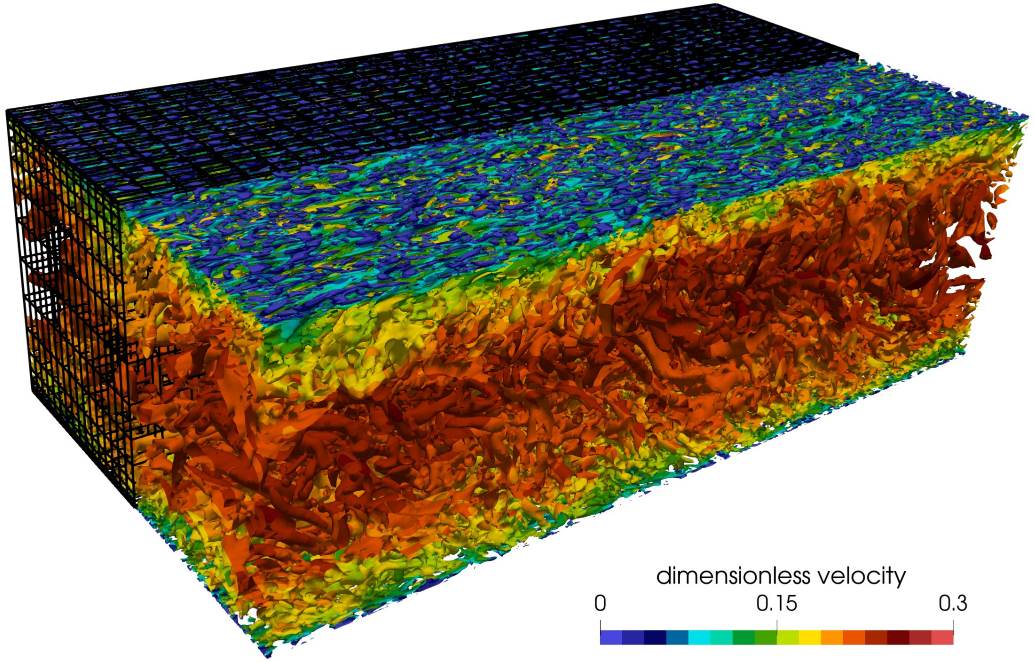


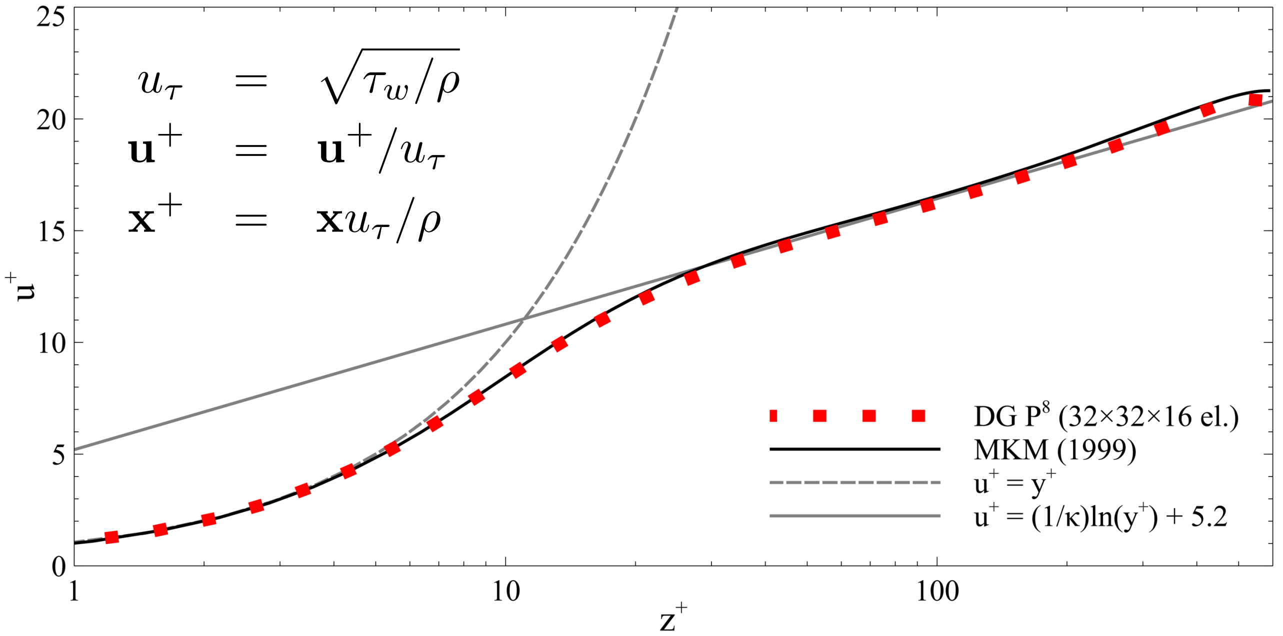


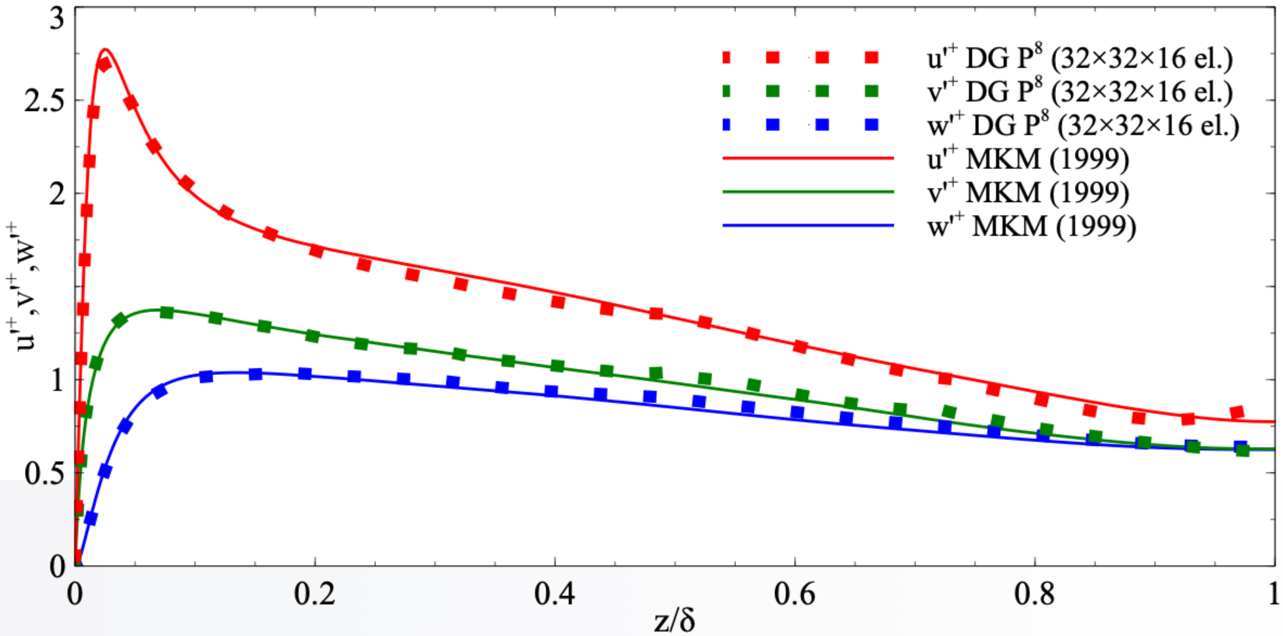
- Domain size: $4\pi \times 2\pi \times 2$
- DG, $k = 8$ - ES, DEEB
- $16\times16\times8$ hexahedral elements
- $\Delta x^+ \approx 26,\ \Delta y^+ \approx 13,\ \Delta z^+ \approx 2$
- RK–SSP 4-5, $\Delta t^+ \approx 0.085$

[BPO] M. Bernardini, S. Pirozzoli, and P. Orlandi, "Velocity statistics in turbulent channel flow up to $Re_\tau = 4\,000$, Journal of Fuid Mechanics, 2014
[MKM] R. D. Moser, J. Kim, and N. N. Mansour, "Direct numerical simulation of turbulent channel flow up to Reτ ≈ 5 590," Physics of Fluids, 1999

implicit-LES - channel flow $Re_\tau = \{180, 395, 590\}$



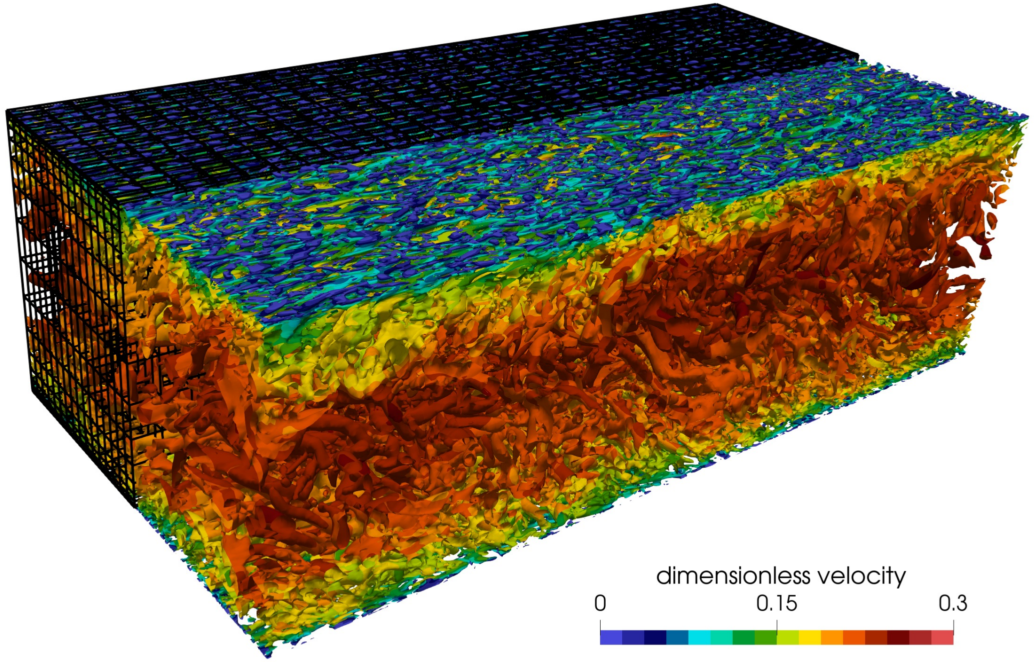


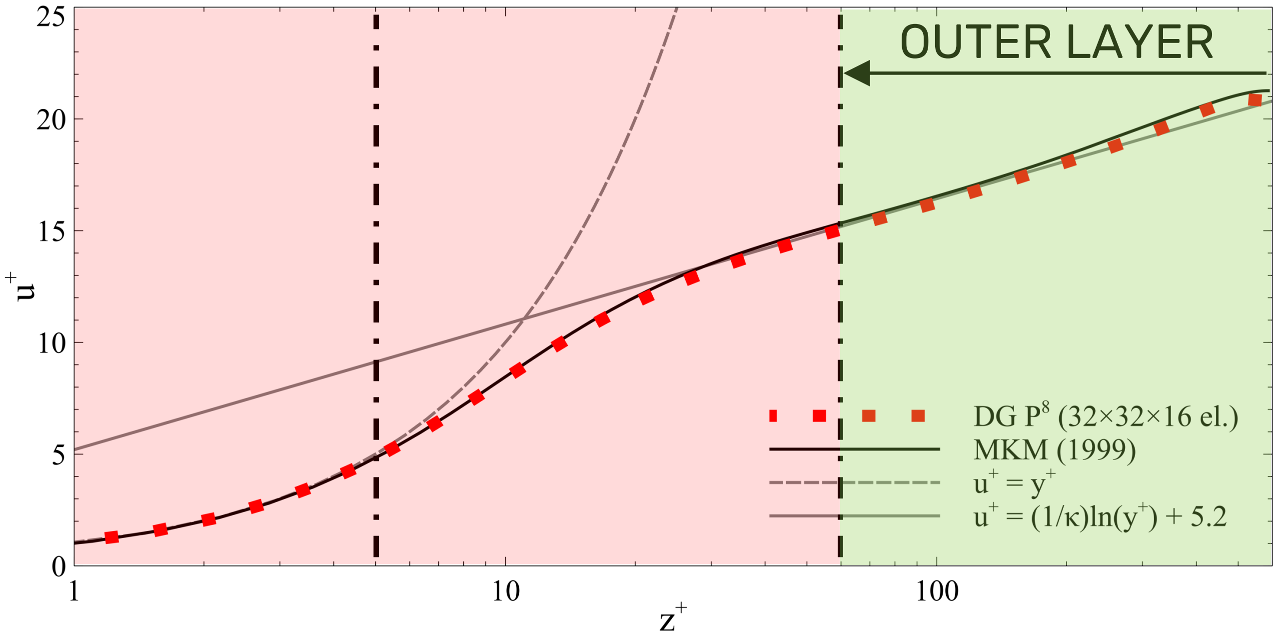


- Domain size: $2\pi \times \pi \times 2$
- DG, $k = 8$ - ES, DEEB
- 32×32×16 hexahedral elements
- $\Delta x^+ \approx 20,\ \Delta y^+ \approx 10,\ \Delta z^+ \approx 2$
- RK–SSP 4-5, $\Delta t^+ \approx 0.006$

[MKM] R. D. Moser, J. Kim, and N. N. Mansour, "Direct numerical simulation of turbulent channel flow up to Reτ ≈ 5 590," Physics of Fluids, 1999

Considerations on the SRS of turbulent flows





1. The high resolution required to capture fine structures in the inner layer is impractical at high Reynolds numbers (~99% of points, for $Re_L = 10^6$ BL *[21]*)
2. Tiny cells at the walls dramatically restrict time step size of explicit time integration
3. Wall-stress models reduce the need for high resolution by replacing the no-slip wall condition with a slip-wall and modelled shear stress at the wall, e.g. *[22,23,24]*.
4. Wall-stress models can reduce the cost for BL turbulence to $N_{wm-LES} \sim Re_L$ *[25]*

[21] U. Piomelli and E. Balaras, "Wall-layer models for large-eddy simulations" Annu. Rev. Fluid Mech. 34, 349–374 (2002)
[22] Frère, A., de Wiart, C. C., Hillewaert, K., Chatelain, P., & Winckelmans, G. (2017). Application of wall-models to discontinuous Galerkin LES. Physics of Fluids, 29.
[23] Hartmann, R. Wall-resolved and wall-modeled ILEs based on high-order DG Proceedings of the 6th European Conference on Computational Mechanics: Solids, Structures and Coupled Problems, ECCM 2018 and 7th European Conference on Computational Fluid Dynamics, ECFD 2018, 2020, pp. 4194–4207
[24] Owen, H., et al. (2020). Wall-modeled large-eddy simulation in a finite element framework. International Journal for Numerical Methods in Fluids, 92(1), 20–37.
[25] Grid-point requirements for large eddy simulation: Chapman's estimates revisited Haecheon Choi; Parviz Moin Physics of Fluids 24, 011702 (2012)

A simple algebraic wall-model approach (wm-iLES) *[26]*

During preprocessing, for each quadrature (destination) point p_d on the wall-modelled face, identify a donor p_D on the opposite cell face, aligned along the normal direction

When assembling the BC contribution to the spatial residual…

1) from the solution at p_D compute $\begin{cases} \mathbf{u}_t = \mathbf{I} - (\mathbf{n} \otimes \mathbf{n})\mathbf{u} \\ \rho, \nu \\ dist\,(p_n, p_d) \end{cases}$

2) solve for u_τ using the Newton method

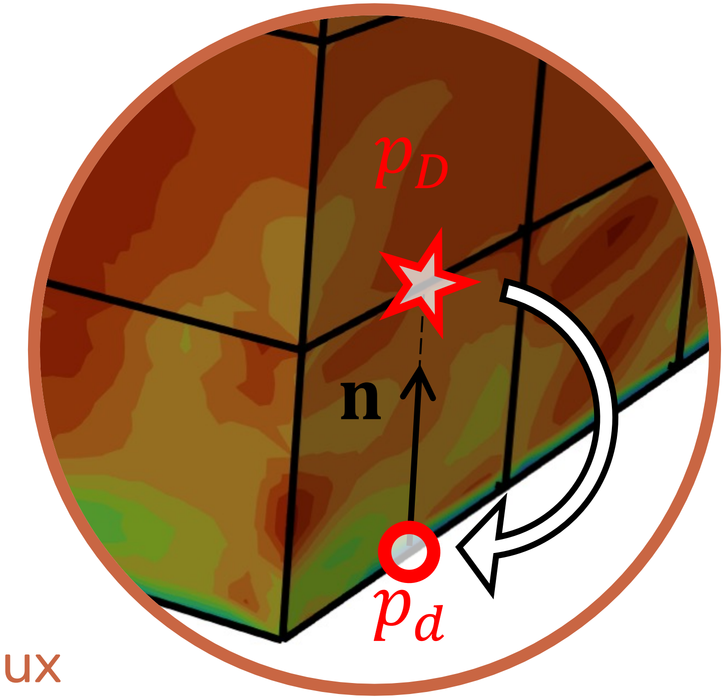
$$G\left(u_\tau\right) = \frac{|\mathbf{u}_t|}{u_\tau} - u^+\left(y^+\right) = 0$$

3) compute the modelled wall shear stress as $\tau_w = \rho u_\tau^2$

4) impose a slip condition $(\mathbf{u} \cdot \mathbf{n} = 0)$ and a modelled viscous flux

$$\hat{\mathbf{F}}_v = \left[0,\ -\tau_w(\mathbf{u}_t/|\mathbf{u}_t|),\ -\tau_w(\mathbf{u}_t/|\mathbf{u}_t|) \cdot \mathbf{n} + \Phi_w\right]^\top$$

where $\Phi_w = 0$ if the wall is adiabatic, or modelled according to Kader *[27]* if isothermal



[26] Hartmann, R. Wall-resolved and wall-modeled ILEs based on high-order DG Proceedings of the 6th European Conference on Computational Mechanics: Solids, Structures and Coupled Problems, ECCM 2018 and 7th European Conference on Computational Fluid Dynamics, ECFD 2018, 2020, pp. 4194–4207
[27] B. Kader, "Temperature and concentration profiles in fully turbulent boundary layers," International Journal of Heat and Mass Transfer, vol. 24, no. 9, pp. 1541–1544, 1981.

A simple algebraic wall-model approach (wm-iLES) *[26]*

During preprocessing, for each quadrature (destination) point p_d on the wall-modelled face, identify a donor p_D on the opposite cell face, aligned along the normal direction

When assembling the BC contribution to the spatial residual…

1) from the solution at p_D compute $\begin{cases} \mathbf{u}_t = \mathbf{I} - (\mathbf{n} \otimes \mathbf{n})\mathbf{u} \\ \rho, \nu \\ dist\,(p_n, p_d) \end{cases}$
2) solve for u_τ using the Newton method considering the Reichardts law-of-the-wall

$$\frac{|\mathbf{u}_t|}{u_\tau} - \frac{1}{\kappa} \ln\left(1 + \kappa y^+\right) + \left(C - \frac{1}{\kappa}\ln(\kappa)\right)\left[1 - e^{-y^+/11} - (y^+/11)e^{-y^+/3}\right] = 0$$

3) compute the modelled wall shear stress as $\tau_w = \rho u_\tau^2$
4) impose a slip condition $(\mathbf{u} \cdot \mathbf{n} = 0)$ and a modelled viscous flux

$$\hat{\mathbf{F}}_v = \left[0,\ -\tau_w(\mathbf{u}_t/|\mathbf{u}_t|),\ -\tau_w(\mathbf{u}_t/|\mathbf{u}_t|) \cdot \mathbf{n} + \Phi_w\right]^\top$$

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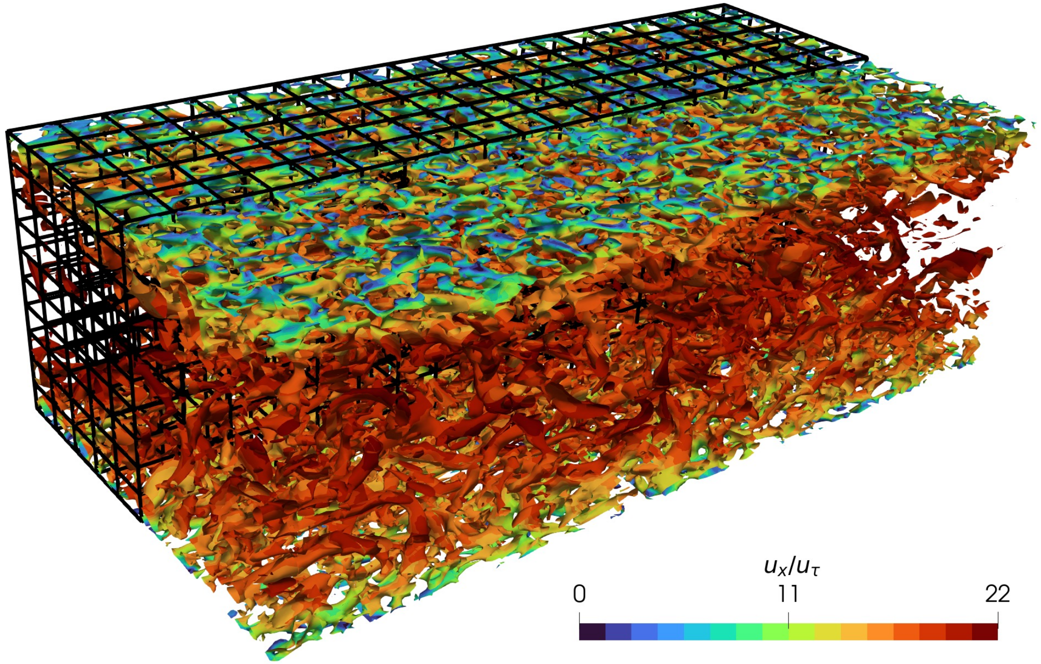
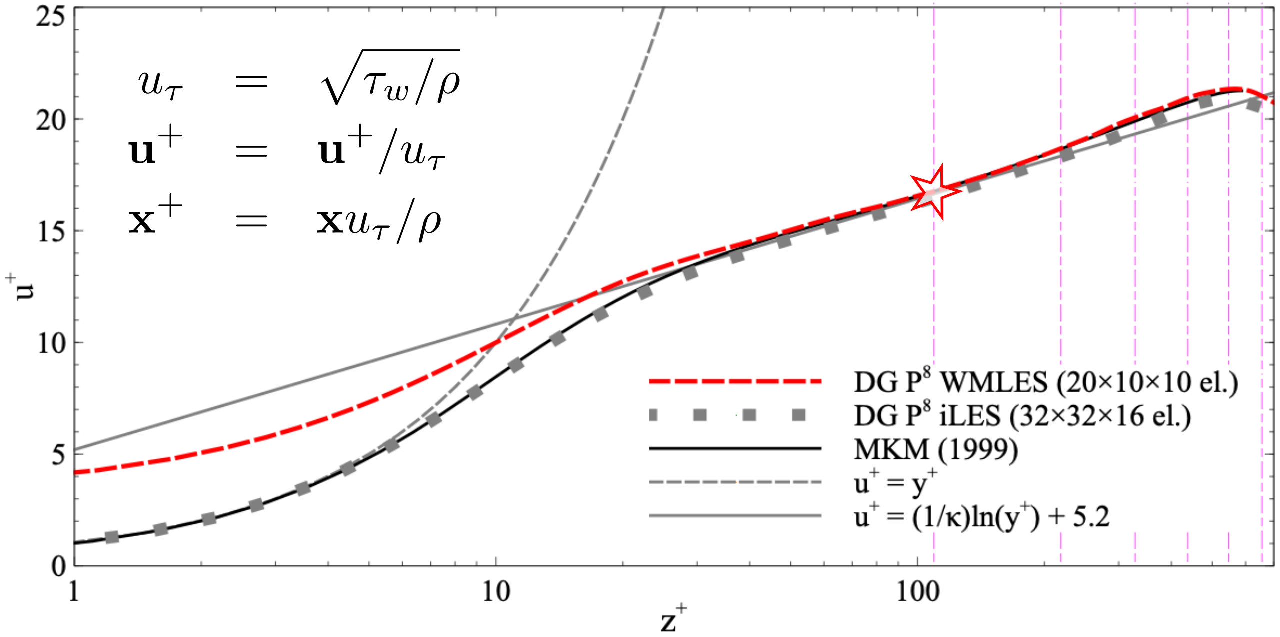
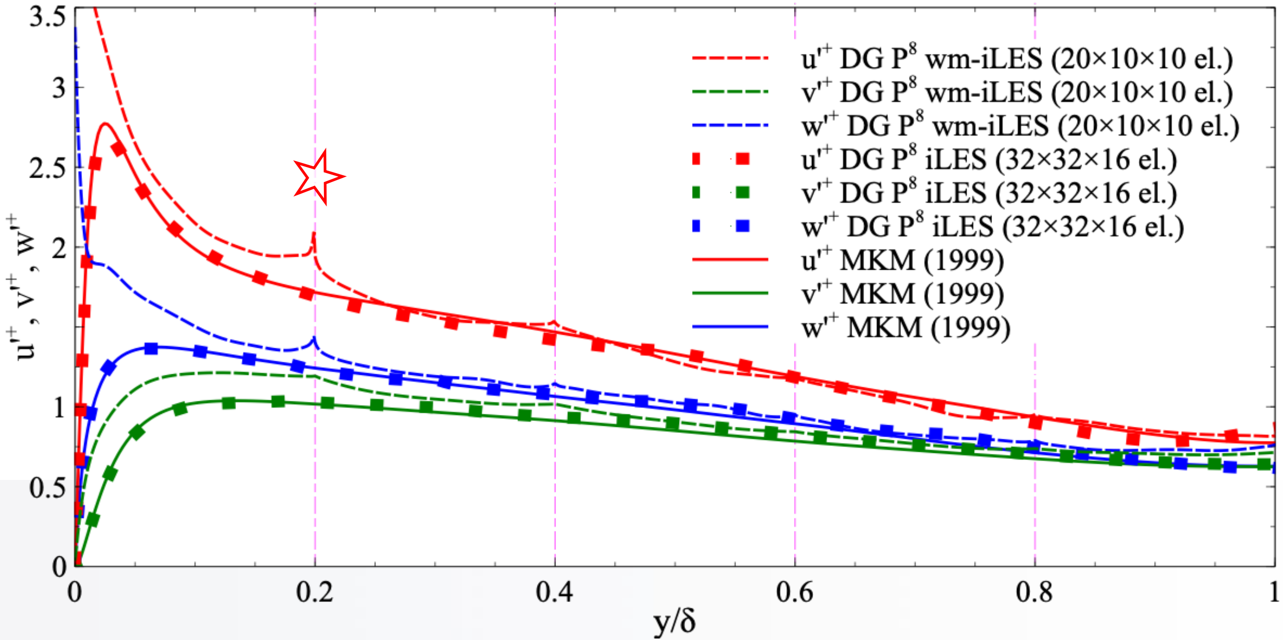
[27] B. Kader, "Temperature and concentration profiles in fully turbulent boundary layers," International Journal of Heat and Mass Transfer, vol. 24, no. 9, pp. 1541–1544, 1981.

Preliminary results - channel flow $\mathrm{Re}_\tau = \{590, 2000\}$



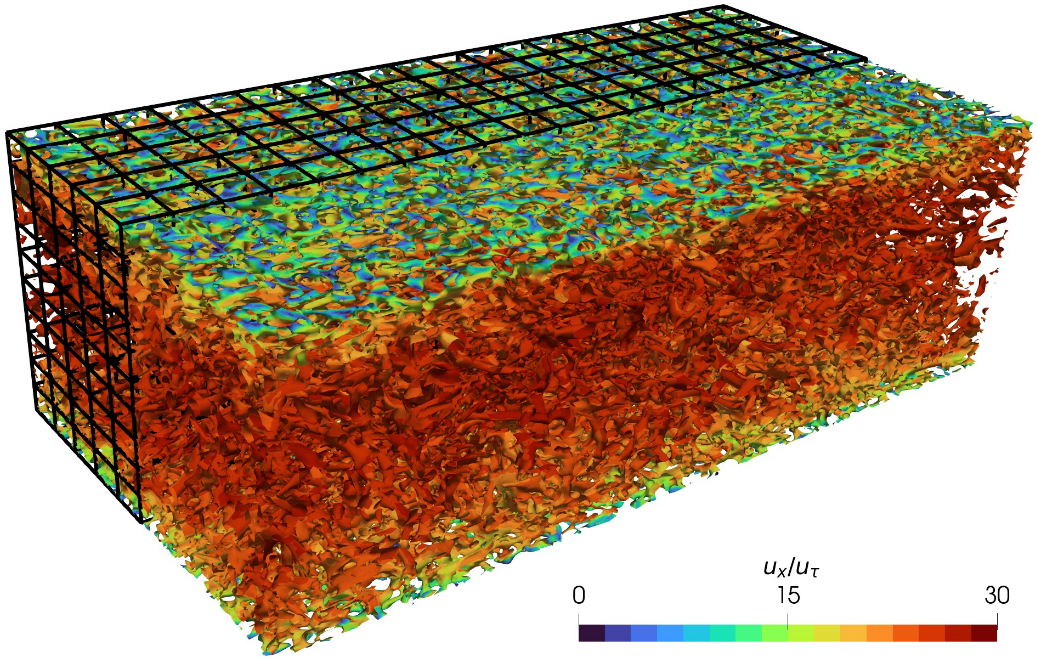


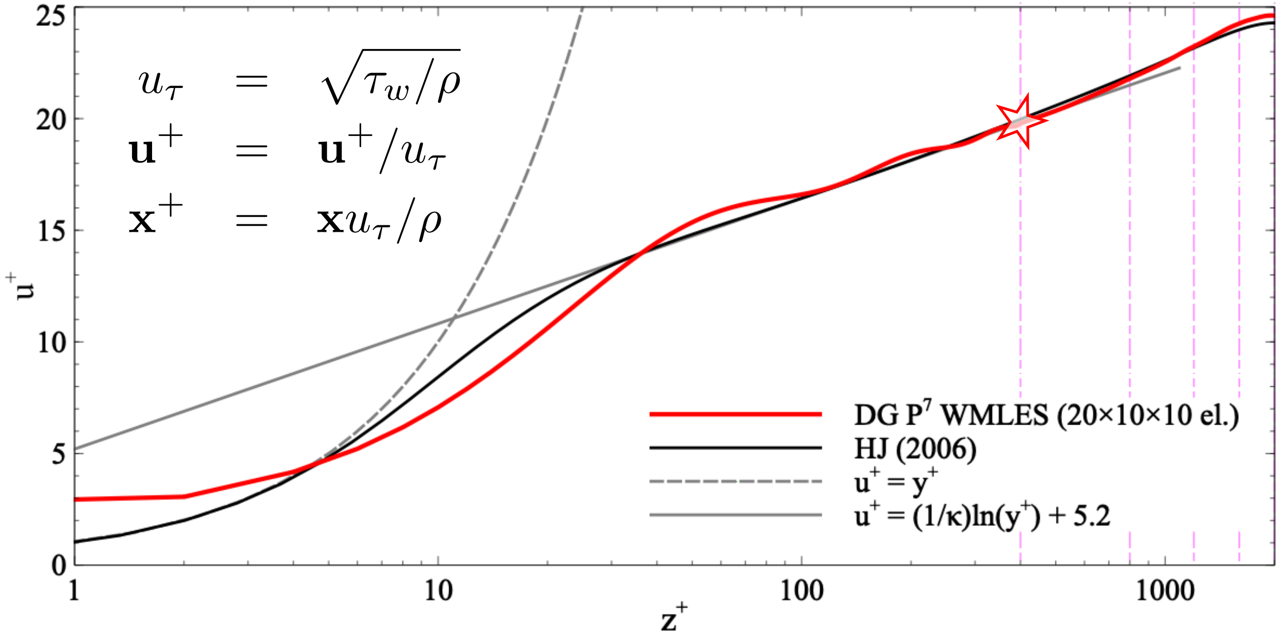


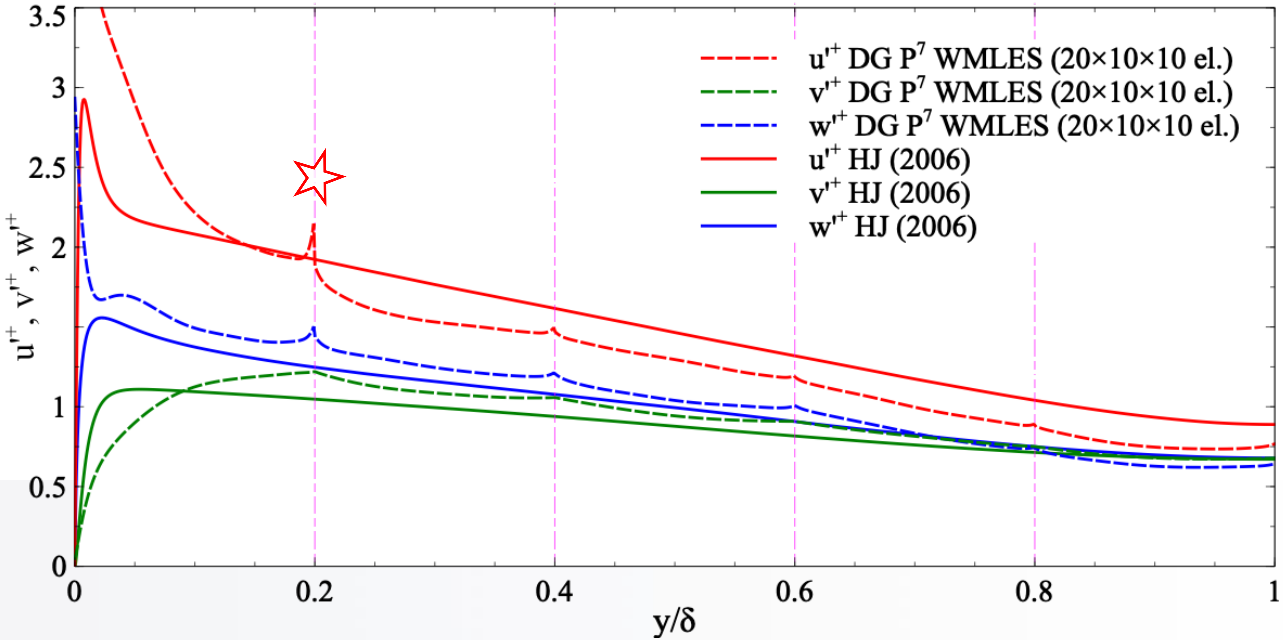
- Domain size: $2\pi \times \pi \times 2$
- DG, $k = 8$ - ES, DEEB
- $20 \times 10 \times 10$ hexahedral elements
- $\Delta x^+ \approx 188,\ \Delta y^+ \approx 188,\ \Delta z^+ \approx 120$
- RK–SSP 4-5, $\Delta t^+ \approx 3.6$

[MKM] R. D. Moser, J. Kim, and N. N. Mansour, "Direct numerical simulation of turbulent channel flow up to Reτ ≈ 5 590," Physics of Fluids, 1999

Preliminary results - channel flow $\mathrm{Re}_\tau = \{590, 2000\}$







- Domain size: $2\pi \times \pi \times 2$
- DG, $k = 7$ - ES, DEEB
- 20×10×10 hexahedral elements
- $\Delta x^+ \approx 628,\ \Delta y^+ \approx 628,\ \Delta z^+ \approx 400$
- RK—SSP 4-5, $\Delta t^+ \approx 4.5$

[HJ] R. D. Moser, J. Kim, and N. N. Mansour, "Direct numerical simulation of turbulent channel flow up to Reτ ≈ 5 590," Physics of Fluids, 1999

Conclusion

We are investigating an efficient explicit discretisation suitable for unstructured meshes and prone to implementation on GPUs

…and future/ongoing work

- considering advanced wall models for iLES, e.g., two-layer models
- extension of the solver to elements of arbitrary shape (polyhedral)
- GPU-porting using OpenACC to approach industrial applications
- incompressible flows simulation via the Entropically Damped Artificial Compressibility formulation