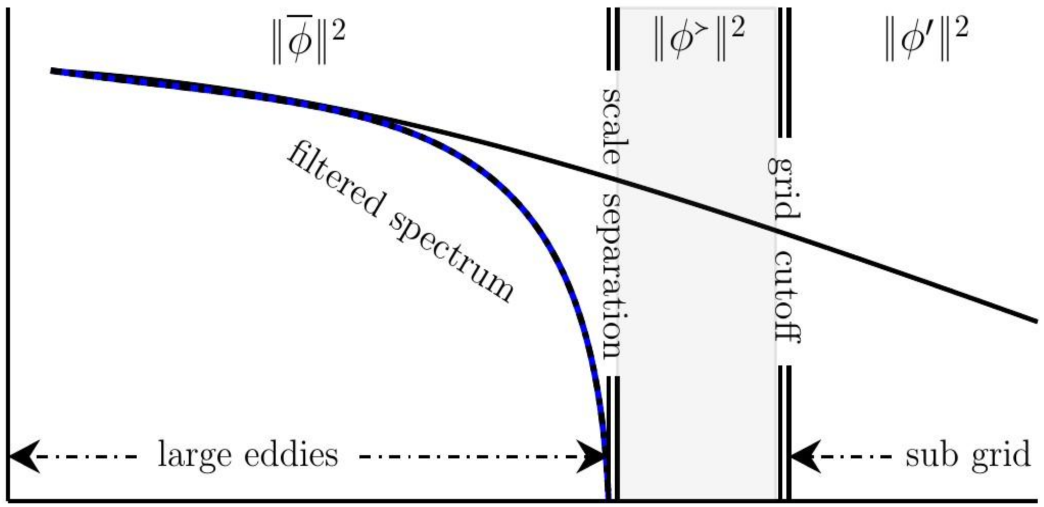
Large Eddy Simulation of Burgulence:

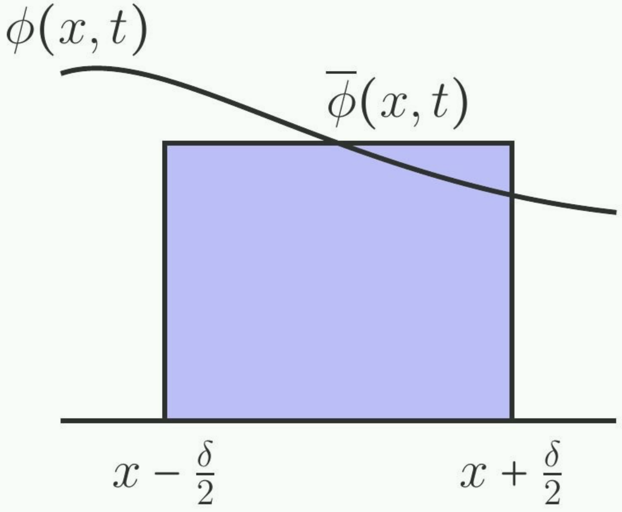
a Synthesis of Filtering, Modelling and Discretization



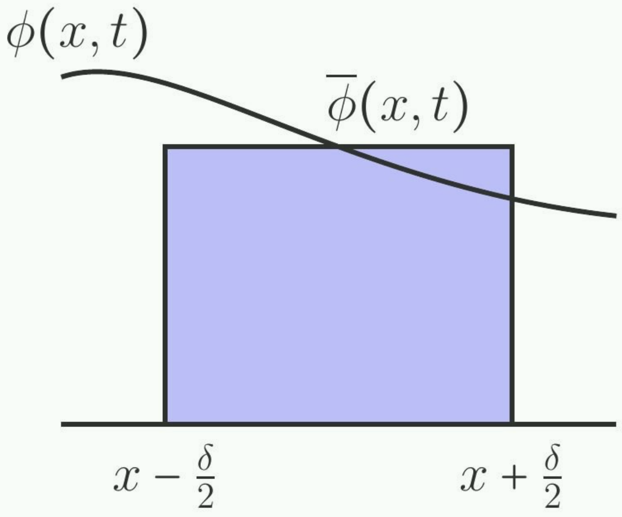
Roel Verstappen

university of groningen

Filter



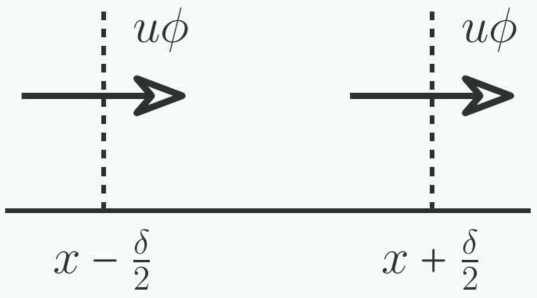
$$\overline{\phi}(x,t) = \frac{1}{\delta} \int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} \phi(x,t)\, dx$$

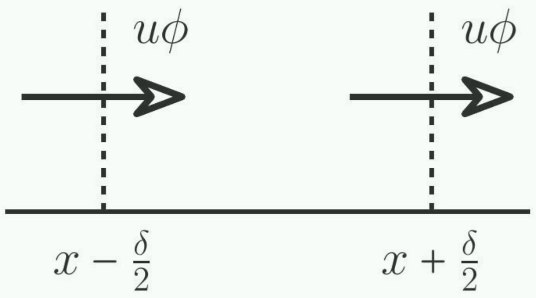


$$\overline{\phi}(x,t) = \frac{1}{\delta}\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} \phi(x,t)\,dx$$

Conservation

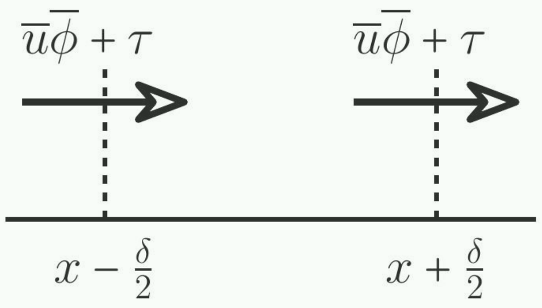
$$\delta\,\partial_t\overline{\phi}(x,t) = (u\phi)(x-\tfrac{\delta}{2},t) - (u\phi)(x+\tfrac{\delta}{2},t)$$





$$\delta\,\partial_t\overline{\phi}(x,t) \;=\; (u\phi)(x-\tfrac{\delta}{2},t) \;-\; (u\phi)(x+\tfrac{\delta}{2},t)$$

Closure

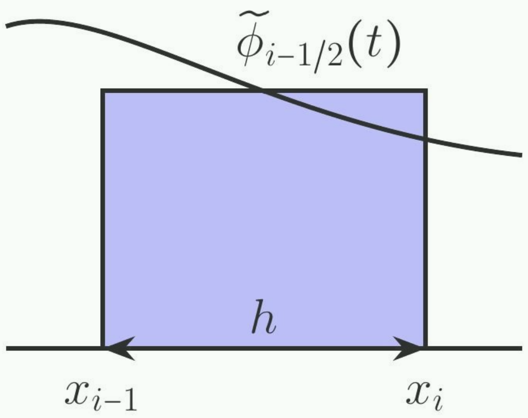


$$\delta\,\partial_t\overline{\phi}(x,t) \;=\; (\overline{u}\overline{\phi}+\tau)(x-\tfrac{\delta}{2},t) \;-\; (\overline{u}\overline{\phi}+\tau)(x+\tfrac{\delta}{2},t)$$

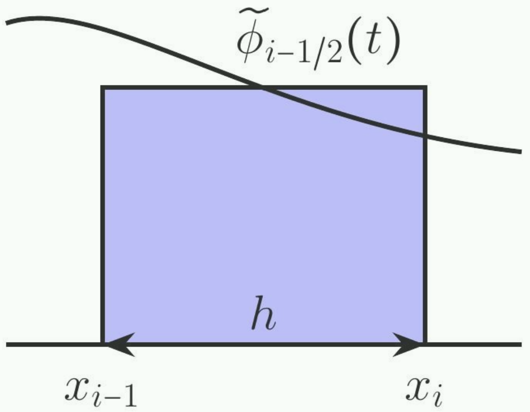
$$\tau(\overline{u},\overline{\phi}) \;\simeq\; u\phi - \overline{u}\overline{\phi}$$

U. Schumann JCP 1975

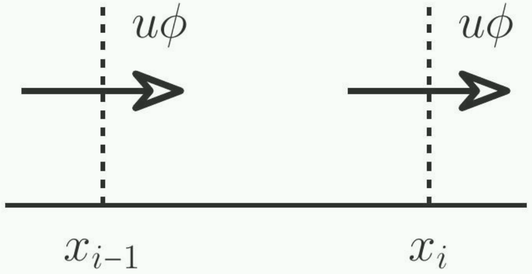
Grid filter



$$\widetilde{\phi}_{i-1/2}(t) = \frac{1}{h} \int_{x_{i-1}}^{x_i} \phi(x,t)\, dx$$

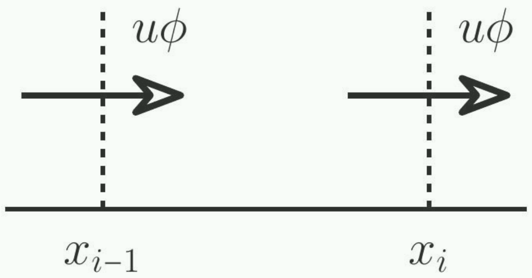


Conservation



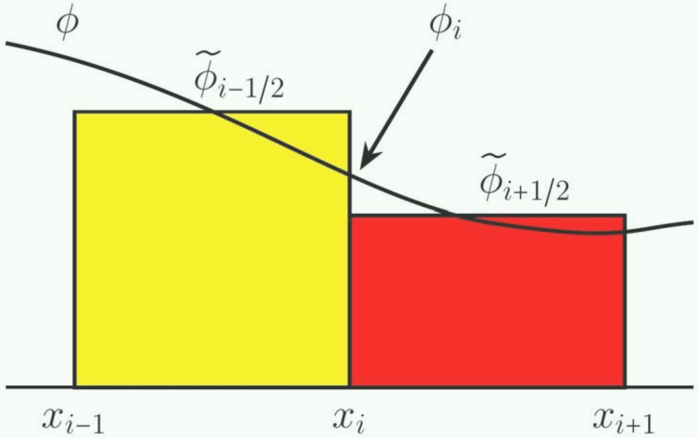
$$\widetilde{\phi}_{i-1/2}(t) = \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(x,t)\,dx$$

$$h\,\partial_t\widetilde{\phi}_{i-1/2} = (u\phi)_{i-1} - (u\phi)_i$$



$$h\,\partial_t\widetilde{\phi}_{i-1/2} = (u\phi)_{i-1} - (u\phi)_i$$

Interpolation



$$\phi_i \simeq \tfrac{1}{2}(\widetilde{\phi}_{i-1/2} + \widetilde{\phi}_{i+1/2})$$

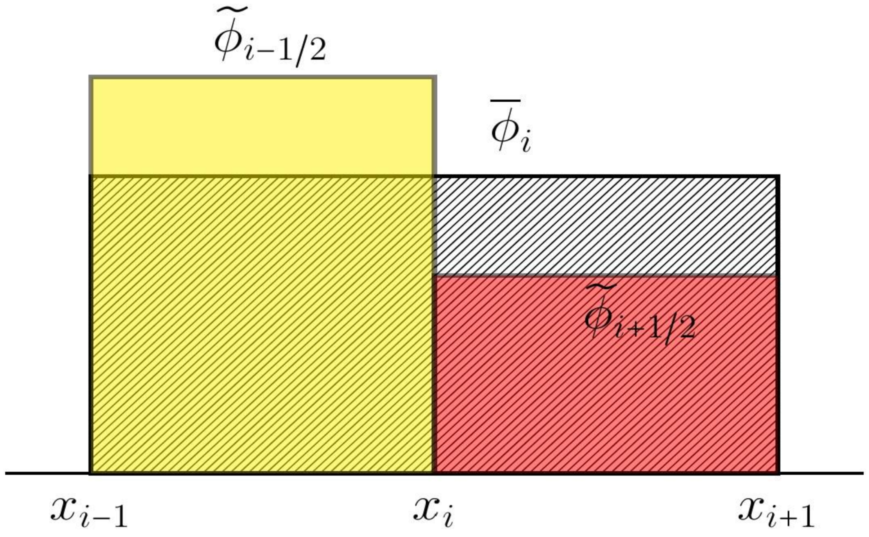
$$= \frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}} \phi(x,t)\,dx = \overline{\phi}_i$$

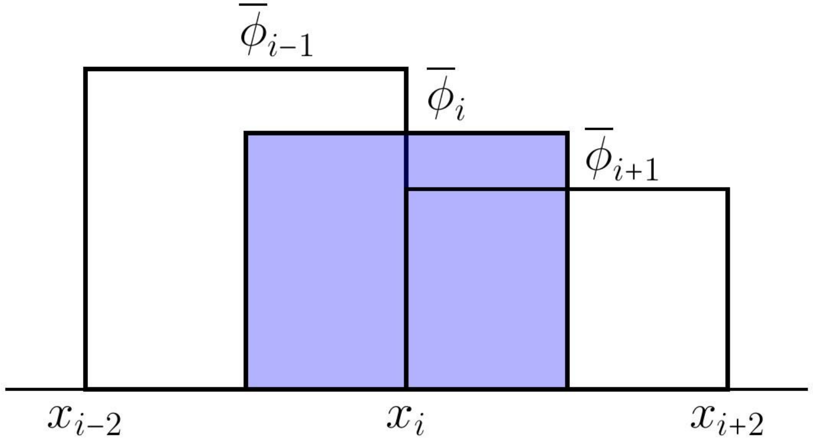
Two filters, lengths h and $\delta = 2h$

Two filters - large eddies are defined by interpolation filter

$$\widetilde{\phi}_{i-1/2}(t) = \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(x,t)\,dx$$

$$\overline{\phi}_i(t) = \frac{1}{\delta}\int_{x_{i-1}}^{x_{i+1}} \phi(x,t)\,dx$$

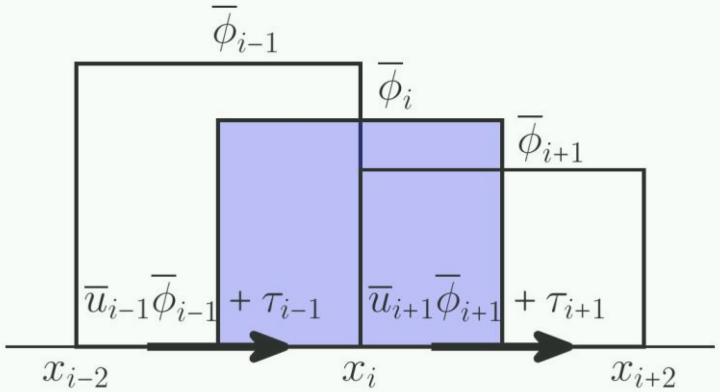




$$\overline{\phi}_i = \tfrac{1}{2}(\widetilde{\phi}_{i-1/2} + \widetilde{\phi}_{i+1/2})$$

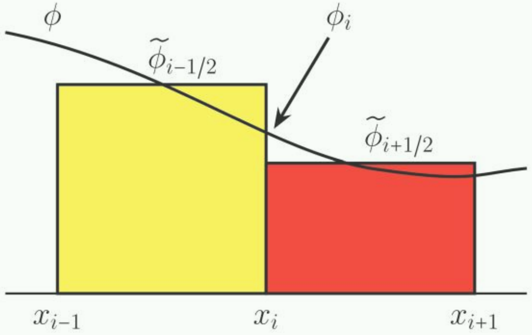
Note: overlapping boxes

FVM



Recall: interpolation

$$\overline{\phi}_i = \tfrac{1}{2}(\widetilde{\phi}_{i-1/2} + \widetilde{\phi}_{i+1/2})$$



Adding neighbours

Truncation error

$$h\,\partial_t\widetilde{\phi}_{i-1/2} + \cancel{\overline{u}_i\overline{\phi}_i} - \overline{u}_{i-1}\overline{\phi}_{i-1} = \cancel{-\tau_i} + \tau_{i-1}$$

$$h\,\partial_t\widetilde{\phi}_{i+1/2} + \overline{u}_{i+1}\overline{\phi}_{i+1} - \cancel{\overline{u}_i\overline{\phi}_i} = -\tau_{i+1} + \cancel{\tau_i} \qquad +$$

$$\delta\,\partial_t\overline{\phi}_i + \overline{u}_{i+1}\overline{\phi}_{i+1} - \overline{u}_{i-1}\overline{\phi}_{i-1} = -\tau_{i+1} + \tau_{i-1}$$

FVM

Truncation

$$\delta\,\partial_t\overline{\phi}_i \quad + \quad \overline{u}_{i+1}\overline{\phi}_{i+1} \quad - \quad \overline{u}_{i-1}\overline{\phi}_{i-1} \quad = \quad -\ \tau_{i+1} \quad + \quad \tau_{i-1}$$

LES

$$\delta\,\partial_t\overline{\phi}(x) + (\overline{u}\overline{\phi})(x+\tfrac{\delta}{2}) - (\overline{u}\overline{\phi})(x-\tfrac{\delta}{2}) \;=\; -\tau(x+\tfrac{\delta}{2}) + \tau(x-\tfrac{\delta}{2})$$

Closure

FVM

$$\delta\, \partial_t \overline{\phi}_i \quad + \quad \overline{u}_{i+1} \overline{\phi}_{i+1} \quad - \quad \overline{u}_{i-1} \overline{\phi}_{i-1} \quad = \quad - \ \tau_{i+1} \quad + \quad \tau_{i-1}$$

Truncation

LES

$$\delta\, \partial_t \overline{\phi}(x) + (\overline{u\phi})(x + \tfrac{\delta}{2}) - (\overline{u\phi})(x - \tfrac{\delta}{2}) \ = -\tau(x + \tfrac{\delta}{2}) + \tau(x - \tfrac{\delta}{2})$$

Closure

No difference between truncation error and closure model if

$$x = x_i \qquad \delta = 2h$$

Box filter

$$\widetilde{\phi}(x,t) \;=\; \frac{1}{h}\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} \phi(\xi,t)\,d\xi$$

commutates with differentiation

$$\partial_x\widetilde{\phi} \;=\; \frac{\phi(x+\frac{h}{2},t)\;-\;\phi(x-\frac{h}{2},t)}{h} \;=\; \widetilde{\partial_x\phi}$$

$$\partial_x \widetilde{\phi} \;=\; \frac{\phi(x+\frac{h}{2},t) \;-\; \phi(x-\frac{h}{2},t)}{h} \;=\; \widetilde{\partial_x \phi}$$

$$\partial_x \widetilde{\widetilde{\phi}} \;=\; \frac{\widetilde{\phi}(x+\frac{h}{2},t) \;-\; \widetilde{\phi}(x-\frac{h}{2},t)}{h}$$

Viscous flux

$$\nu\,\partial_x \phi \;=\; \nu\,\partial_x \widetilde{\widetilde{\phi}} \;+\; \nu\,\partial_x(\phi - \widetilde{\widetilde{\phi}})$$

$$=\; \nu\,\frac{\widetilde{\phi}(x+\frac{h}{2},t) \;-\; \widetilde{\phi}(x-\frac{h}{2},t)}{h} \;+\; \nu\,\partial_x(\phi - \widetilde{\widetilde{\phi}})$$

unclosed term -> τ

subgrid modeling and discretization are intrinsically linked

Summary Flux

Filters

$$\widetilde{\phi}_{i-1/2}(t) = \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(x,t)\,dx \qquad \overline{\phi}_i(t) = \frac{1}{\delta}\int_{x_{i-1}}^{x_{i+1}} \phi(x,t)\,dx$$

Flux

$$\overline{u}_i\overline{\phi}_i - \nu\,\frac{\widetilde{\phi}_{i+1/2} - \widetilde{\phi}_{i-1/2}}{h} + \tau_i$$

LES model

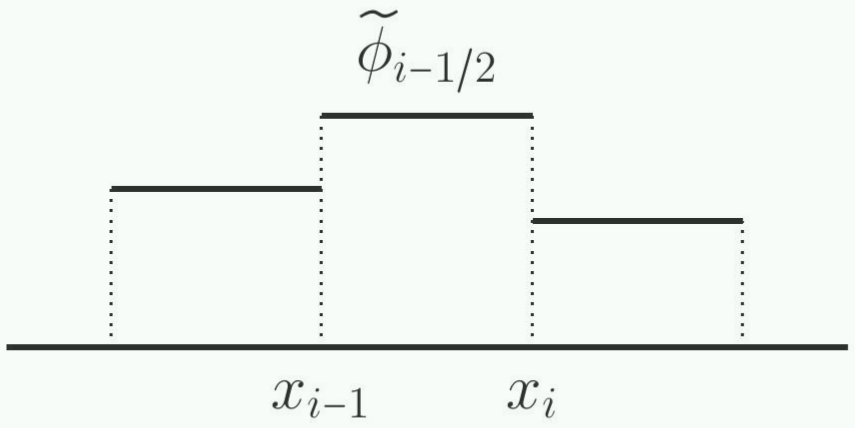
error FVM

Grid filter

$$\widetilde{\phi}(x,t) = \widetilde{\phi}_{i-1/2}(t) \quad for \quad x_{i-1} \leq x < x_i$$

Idempotent

$$\widetilde{\widetilde{\phi}} = \widetilde{\phi} \qquad \Longrightarrow \qquad \widetilde{\phi'} = \widetilde{\phi - \widetilde{\phi}} = 0$$



Grid filter

$$\Longrightarrow \qquad \widetilde{\phi'} = 0$$

Symmetric

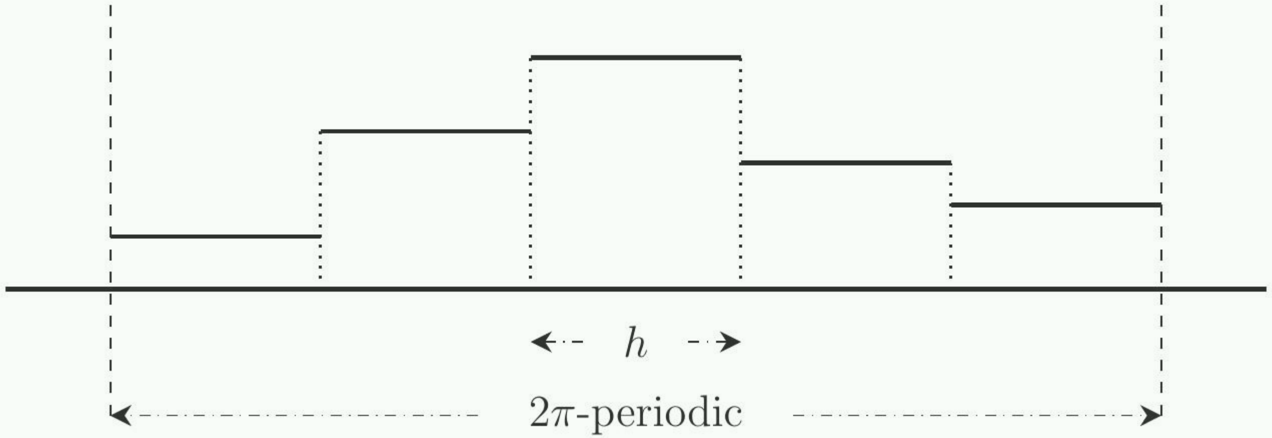
$$(\widetilde{\phi}, \phi') = (\phi, \widetilde{\phi'}) = 0$$

$$(\widetilde{\psi}, \phi') = (\psi, \widetilde{\phi'}) = 0$$

Orthogonal decomposition of function space

Finite volume space

$$\widetilde{\mathcal{F}}_h(0,2\pi)$$



$$\widetilde{\phi}(x,t) \in \widetilde{\mathcal{F}}_h(0,2\pi) \quad \Longrightarrow \quad \widetilde{\phi}(x \pm h,t) \in \widetilde{\mathcal{F}}_h(0,2\pi)$$

$$\widetilde{u} \in \widetilde{\mathcal{F}}_h(0,2\pi) \;\; and \;\; \widetilde{\phi} \in \widetilde{\mathcal{F}}_h(0,2\pi) \quad \Longrightarrow \quad \widetilde{u}\widetilde{\phi} \in \widetilde{\mathcal{F}}_h(0,2\pi)$$

$$\widetilde{u} \in \widetilde{\mathcal{F}}_h(0,2\pi) \;\; and \;\; \widetilde{\phi} \in \widetilde{\mathcal{F}}_h(0,2\pi) \quad \Longrightarrow \quad \partial_t\widetilde{\phi} \in \widetilde{\mathcal{F}}_h(0,2\pi)$$

No subgrid scales are produced

$$\partial_t \widetilde{\phi} \in \widetilde{\mathcal{F}}_h(0, 2\pi) \quad \Longrightarrow \quad (\partial_t \widetilde{\phi}, \phi') = 0$$

$$\Longrightarrow \quad d_t ||\phi'||^2 = 0$$

$$d_t ||\phi||^2 = d_t ||\widetilde{\phi}||^2 + d_t ||\phi'||^2 = d_t ||\widetilde{\phi}||^2$$

Large and small supergrid scales

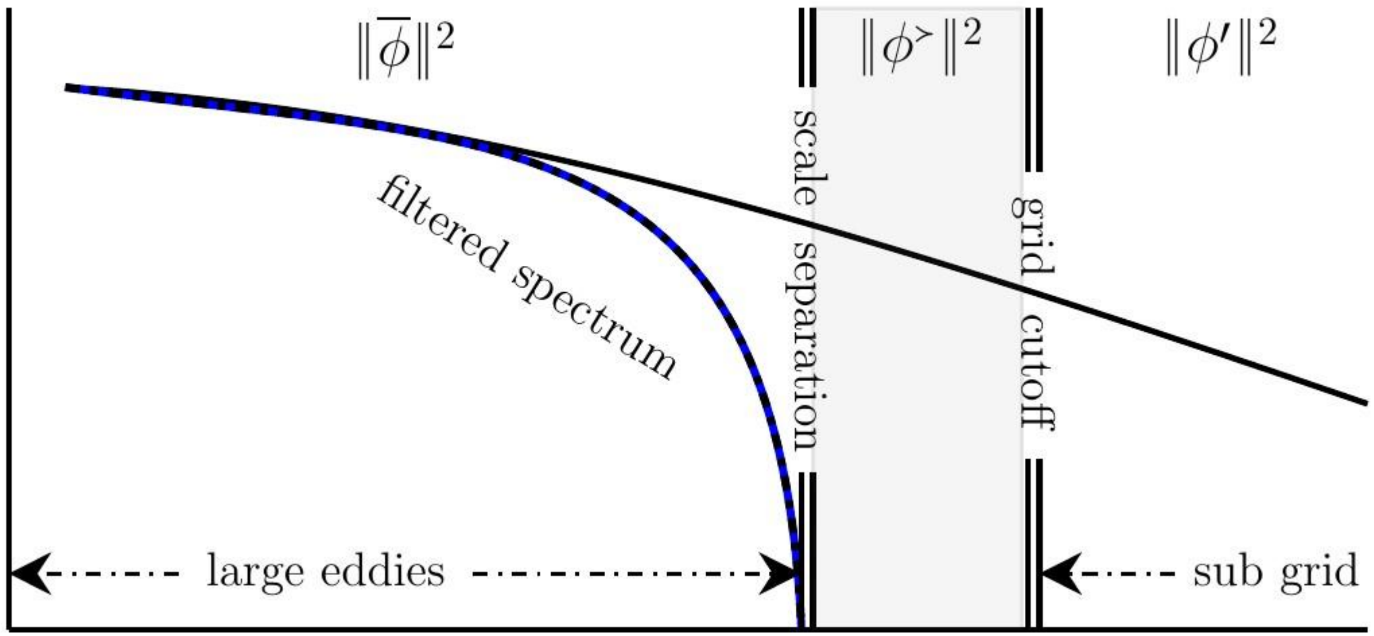
$$\widetilde{\phi}_{i-1/2} = \underbrace{\tfrac{1}{2}(\widetilde{\phi}_{i+1/2} + \widetilde{\phi}_{i-1/2})}_{\overline{\phi}_i} + \underbrace{\tfrac{1}{2}(\widetilde{\phi}_{i-1/2} - \widetilde{\phi}_{i+1/2})}_{\phi_i^{\succ}}$$

Orthogonal decomposition $\quad \overline{\boldsymbol{\phi}} \cdot \boldsymbol{\phi}^{\succ} = 0$

Dissipation $\quad d_t||\boldsymbol{\phi}||^2 = d_t||\widetilde{\boldsymbol{\phi}}||^2 = d_t||\overline{\boldsymbol{\phi}}||^2 + d_t||\boldsymbol{\phi}^{\succ}||^2$

Dissipation condition

$$d_t\|\boldsymbol{\phi}\|^2 = d_t\|\overline{\boldsymbol{\phi}}\|^2 + \cancel{d_t\|\boldsymbol{\phi}^{\succ}\|^2}$$



Small supergrid scales

$$\phi_i^{\succ} = \frac{1}{2}\left(\widetilde{\phi}_{i-1/2} - \widetilde{\phi}_{i+1/2}\right) \qquad \boldsymbol{\phi}^{\succ} = \boldsymbol{S}\widetilde{\boldsymbol{\phi}}$$

$$\partial_t \widetilde{\boldsymbol{\phi}} = \boldsymbol{\nabla}^\top \underbrace{(\boldsymbol{\Phi} + \boldsymbol{\tau})}_{\text{flux}} \qquad \partial_t \boldsymbol{\phi}^{\succ} = \underbrace{\boldsymbol{S}\boldsymbol{\nabla}^\top}_{\boldsymbol{\nabla}_s^\top}(\boldsymbol{\Phi} + \boldsymbol{\tau})$$

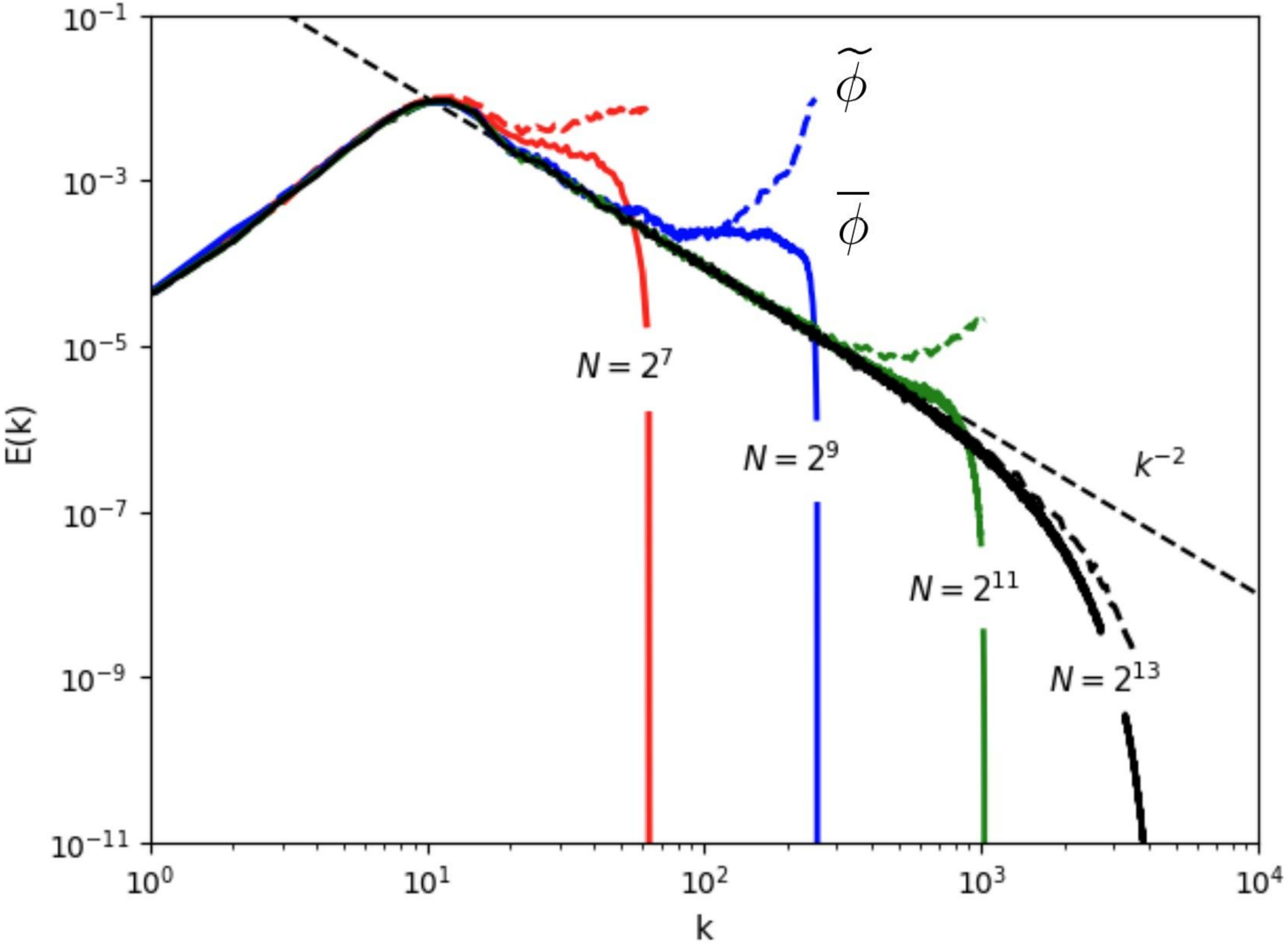
Model

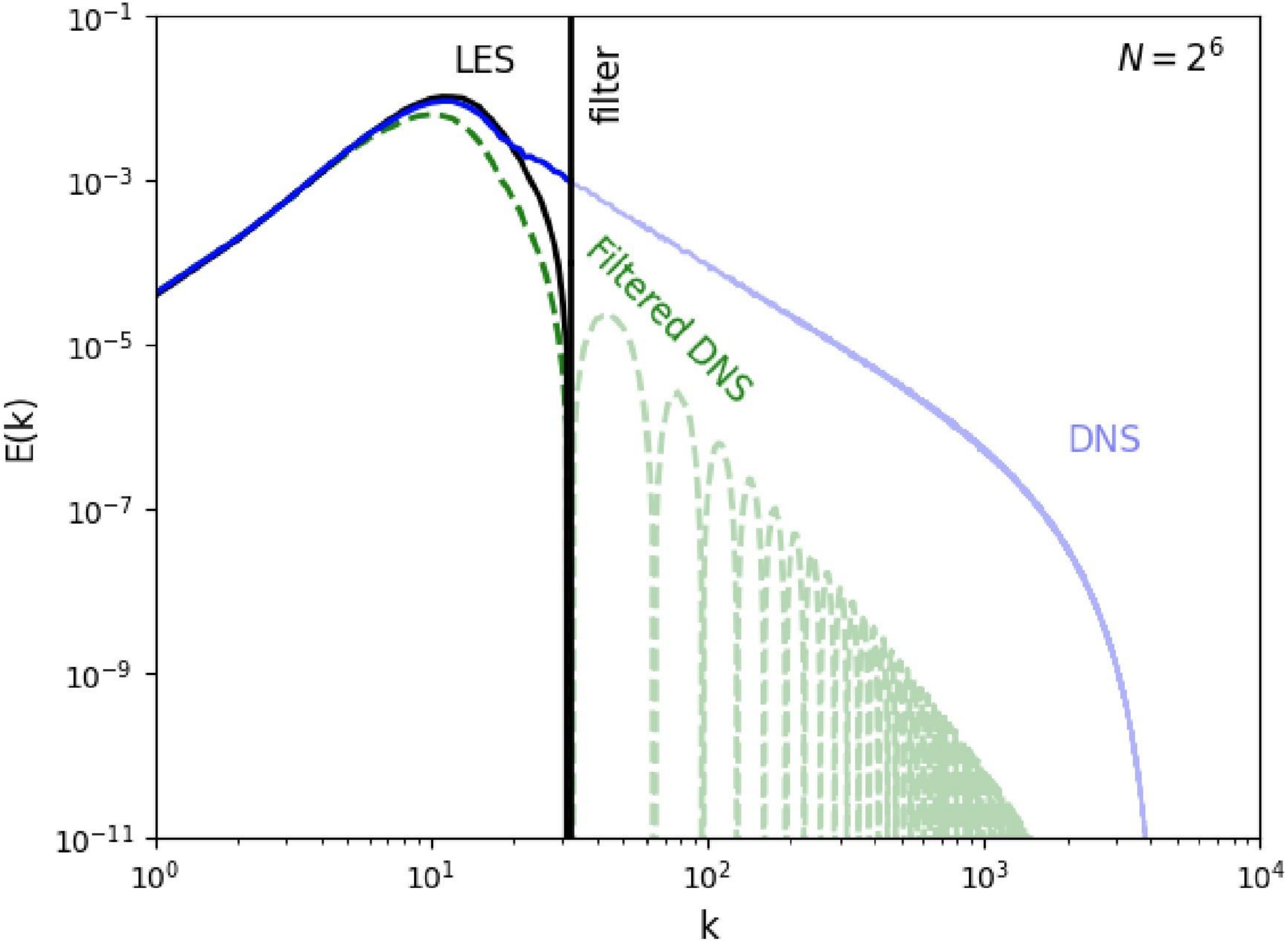
$$\partial_t \boldsymbol{\phi}^{\succ} \quad = \quad \nabla_s^{\top}\big(\boldsymbol{\Phi} + \boldsymbol{\tau}\big)$$

$$d_t \boldsymbol{\phi}^{\succ} \cdot \boldsymbol{\phi}^{\succ} \quad = \quad \big(\boldsymbol{\Phi} + \boldsymbol{\tau}\big) \cdot \nabla_s \boldsymbol{\phi}^{\succ} \quad = \quad 0$$

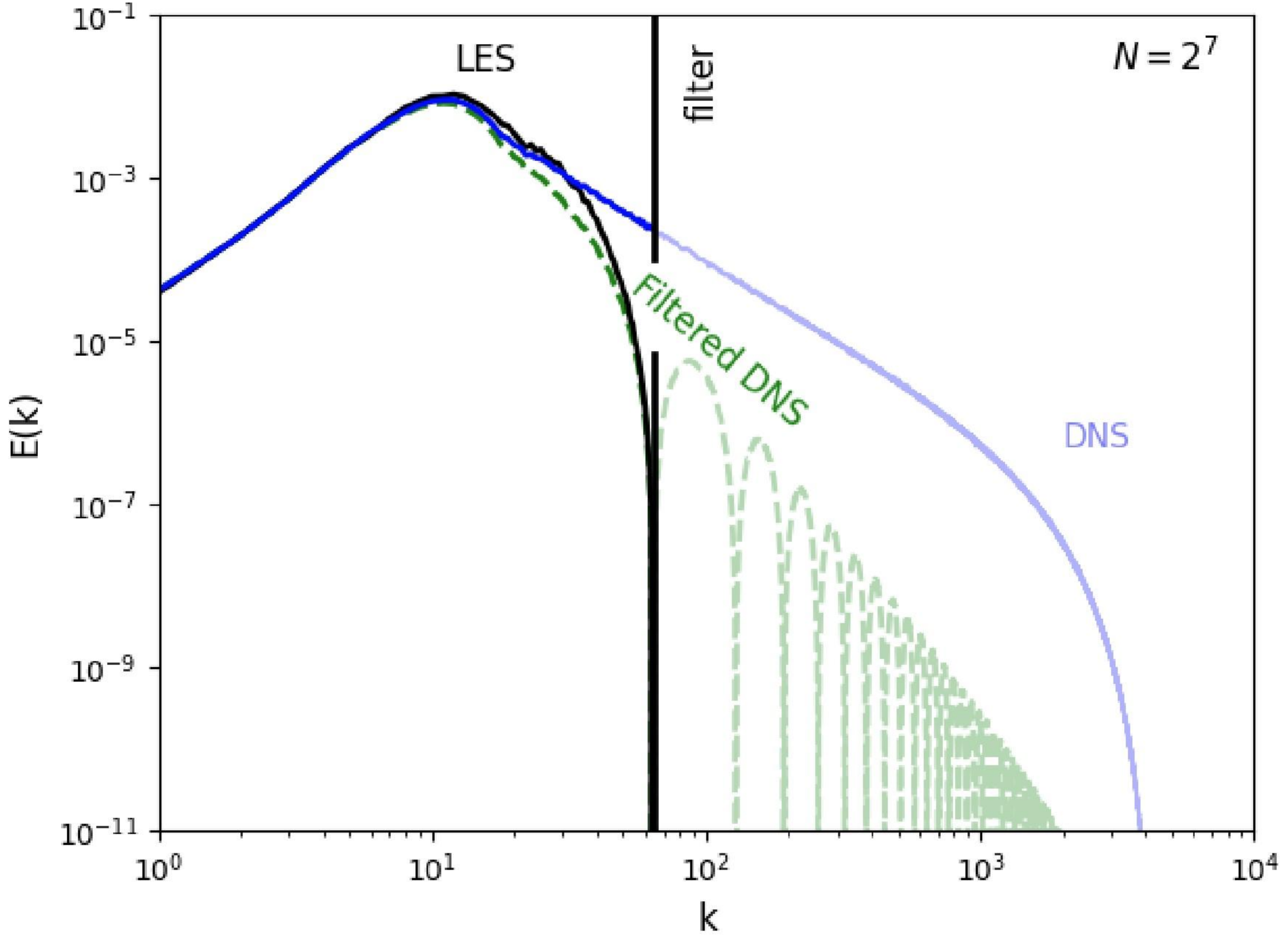
$$\boldsymbol{\tau} \quad = \quad -\alpha \, \nabla_s \boldsymbol{\phi}^{\succ} \qquad\qquad \alpha \quad = \quad \frac{\big(\nabla_s \phi^{\succ} \cdot \boldsymbol{\Phi}\big)}{\big\|\nabla_s \phi\big\|^2}$$

Decaying Burgers' turbulence without model

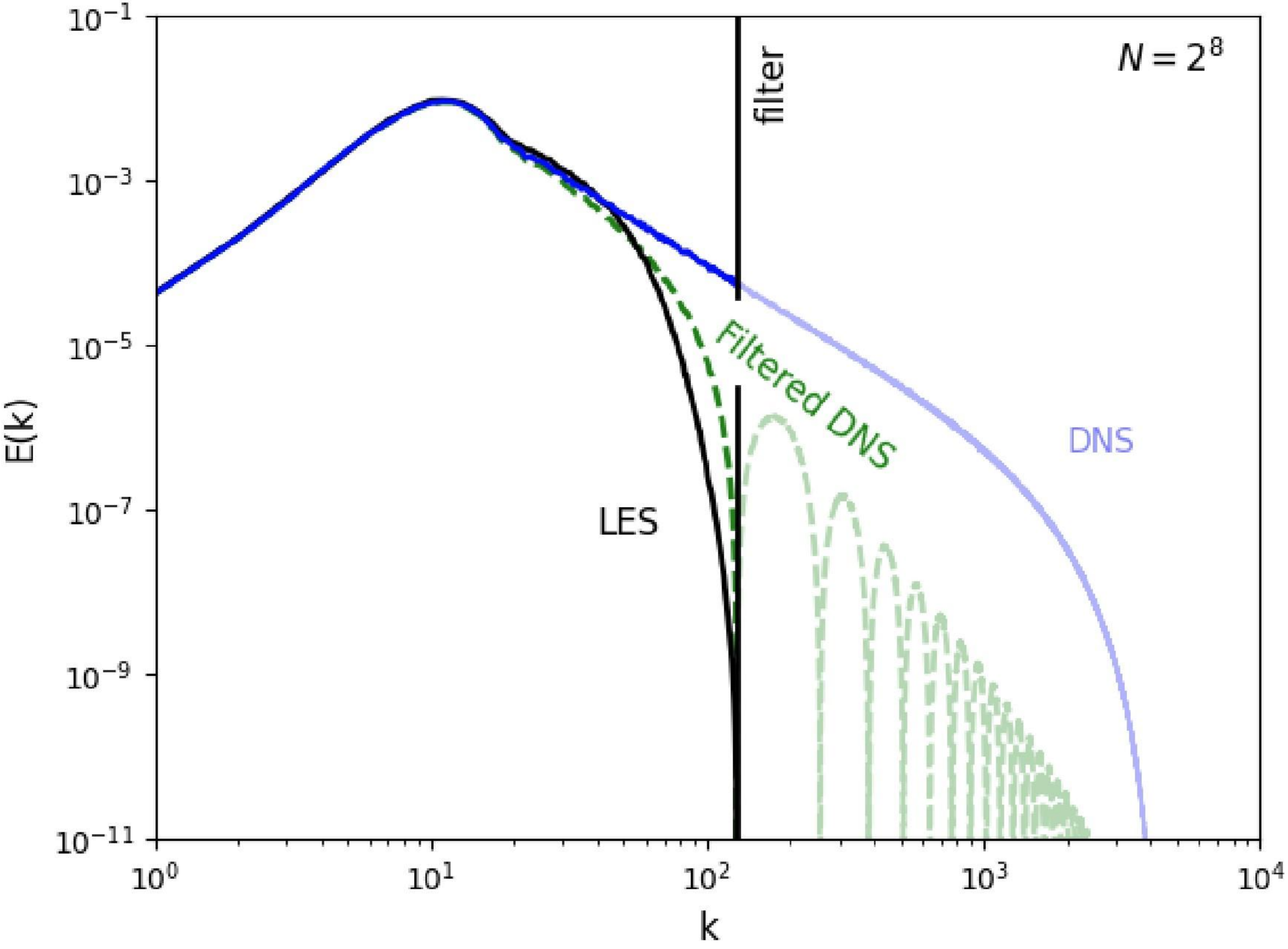




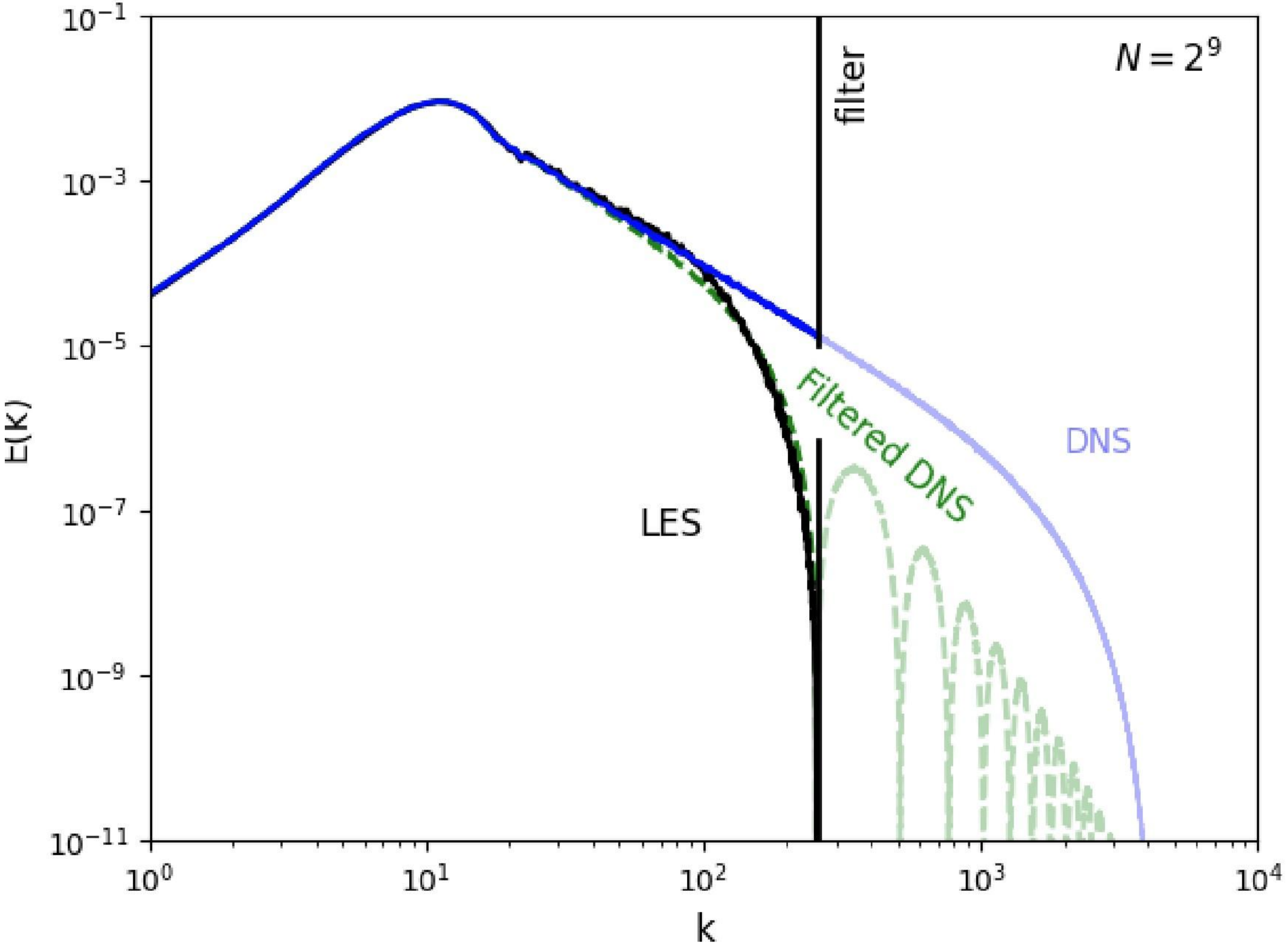
LES
filter
$N = 2^6$
Filtered DNS
DNS
E(k)
k
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^0
10^1
10^2
10^3
10^4



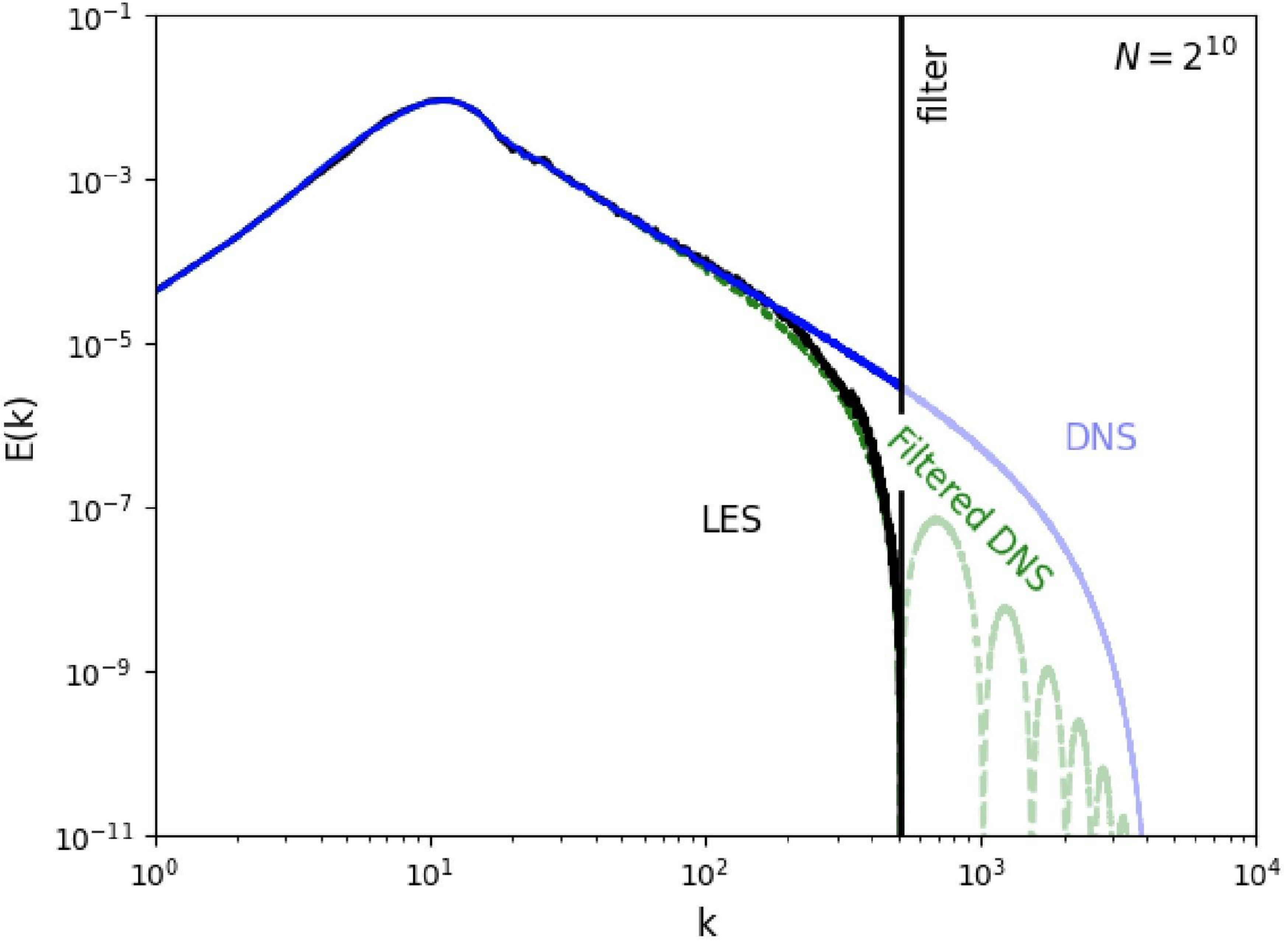
LES
filter
$N = 2^7$
Filtered DNS
DNS
E(k)
k
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}



$N = 2^8$
filter
Filtered DNS
DNS
LES
E(k)
k
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^0
10^1
10^2
10^3
10^4



10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
E(k)
10^0
10^1
10^2
10^3
10^4
k
$N = 2^9$
filter
Filtered DNS
DNS
LES

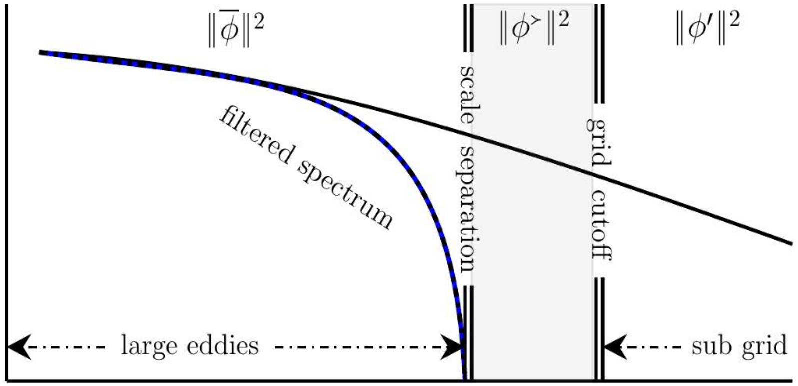


$N = 2^{10}$
filter
DNS
Filtered DNS
LES
E(k)
k
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}

Concluding remarks

FVM-based LES uses two filters that divide the energy into three pieces

Actually no difference between
physical and numerical model -> merge



Model successfully applied
to decaying Burgers' turbulence

To do: 1D -> 3D, nonuniform grids, etc.

Merging Filtering, Modeling and Discretization to Simulate Large Eddies in Burgers' Turbulence

Roel Verstappen

Bernouilli Institute, University of Groningen, Nijenborgh 9, 9747AG Groningen, Netherlands

r.w.c.p.verstappen@rug.nl